Alexander Polynomial the Great

Alycia Doucette and Elizabeth Benda - Mentored by Melody Molander

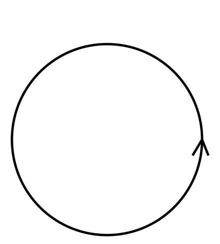
2022 Mathematics Directed Reading Program. University of California - Santa Barbara

Introduction

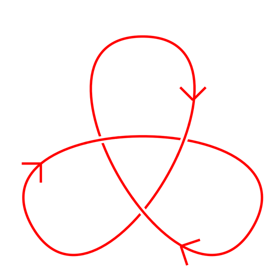
What is a **knot**? Simply speaking, a knot is a closed curve in space that does not intersect itself in any way. Knots have many applications to other fields of science and are fun for mathematicians to study. One of the main questions posed when studying knots is how to tell whether or not two different projections are the same knot. A tool that has developed as a way to distinguish two knots from each other is representing knots as polynomials. In this poster we will focus on one of the three major polynomial representations of knots, the Alexander polynomial.

Definitions

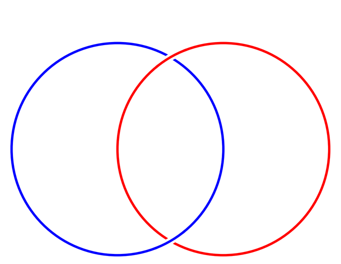
- **Projection:** A two-dimensional picture representation of a knot.
- **Orientation:** A direction in which you travel around the knot.
- **Crossing number:** The least number of crossings that occur in any projection of a particular knot.
- **Link:** A set of knotted loops tangled up together.
- **Unknot:** The unknot is also known as the trivial knot, and it looks as follows:



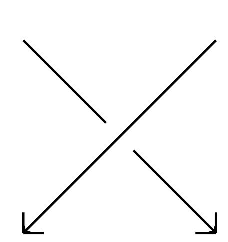
(a) Hi, I'm an oriented unknot!



(b) Hi, I'm an oriented trefoil!



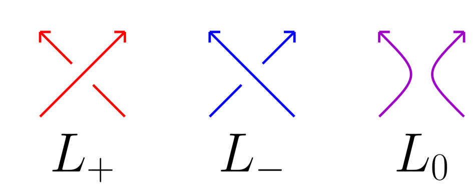
(c) Hi, I'm a link!



(d) Hi, I'm a crossing!

The Alexander Polynomial

The Alexander polynomial was a method invented in 1928 as a way to represent knots and links as polynomial equations. It is an invariant for all representations of knots and links up to the same orientation. The Alexander polynomial is dependent on the orientation of the knot or link being assessed. The formula to compute the Alexander polynomial was refined by John Conway in 1969, and is now based on the following two rules:

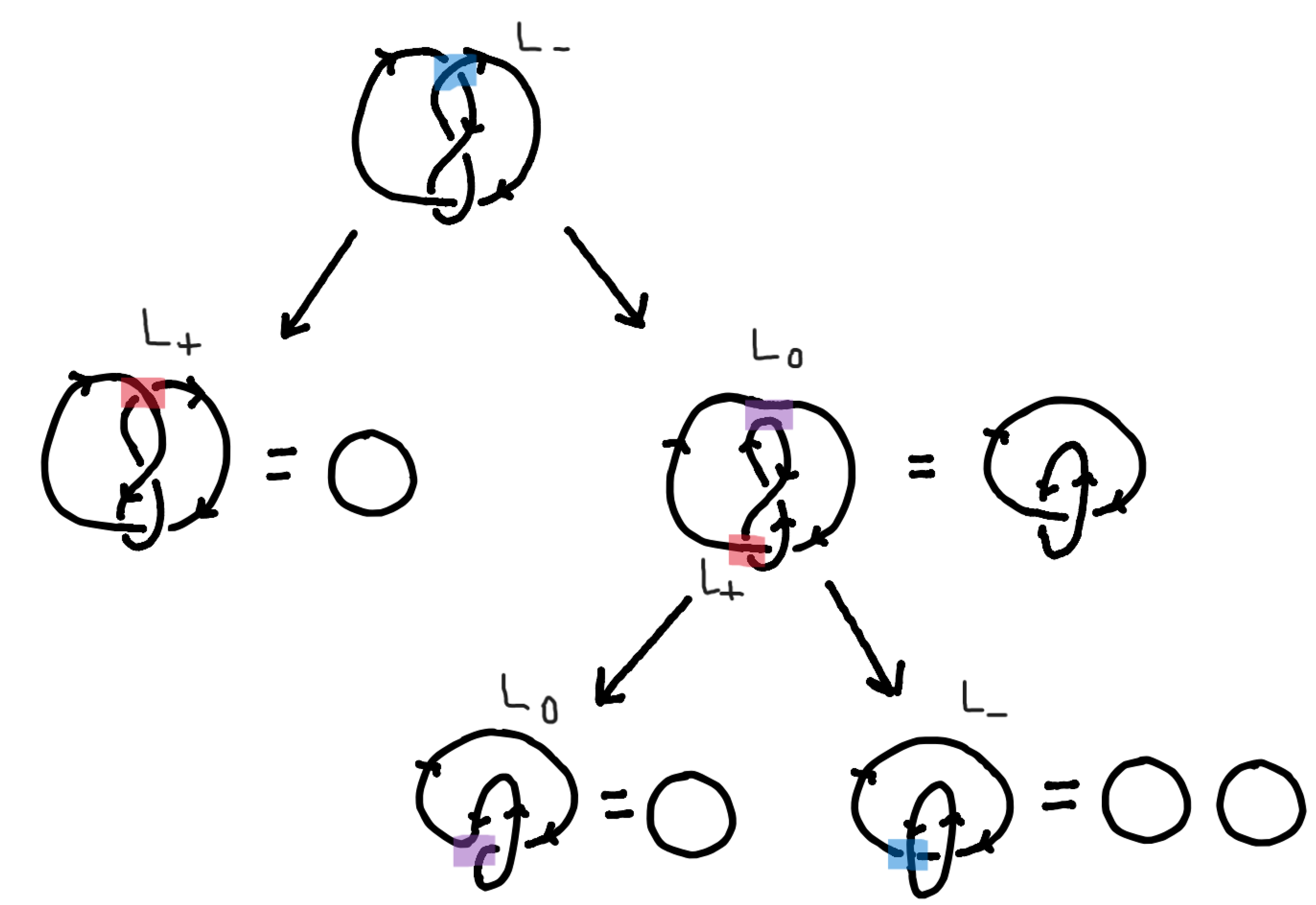


$$\Delta(\bigcirc) = 1 \quad (1)$$

$$\Delta(L_+) - \Delta(L_-) + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\Delta(L_0) = 0 \quad (2)$$

The main tool used to compute the Alexander polynomial is called the **resolving tree**. The resolving tree is an easy way to break a knot down into a series of unknots and trivial links. In order to create the resolving tree, you choose one crossing of the knot, and determine whether it is an L_+, L_-, or L_0 crossing. From there, the chosen crossing is broken down into two new knots. These new knots are dependent on what type of crossing the original one is.

Resolving Tree of the Figure-Eight Knot



Alexander Polynomial of the Figure-Eight Knot

$$\Delta(L_+) - \Delta(L_-) + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\Delta(L_0) = 0$$

$$\Delta(L_+) = \Delta(\bigcirc) = 1$$

$$\Delta(L_0) = \Delta(\bigcirc \cup \bigcirc) - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\Delta(\bigcirc) = -(t^{\frac{1}{2}} - t^{-\frac{1}{2}})$$

$$\Rightarrow \Delta(L_-) = \Delta(L_+) + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\Delta(L_0) = 1 + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})(-t^{\frac{1}{2}} + t^{-\frac{1}{2}})$$

$$= \boxed{3 - t - t^{-1}}$$

Since $3 - t - t^{-1} \neq 1$ we know that the figure-eight knot is not a projection of the unknot.

Other Polynomial Representations

The other polynomial representations we looked at were the Jones polynomial and the HOMFLY polynomial. The Jones polynomial, $V(t)$, is derived using three rules, and the base variable $t^{\frac{1}{2}}$. All prime knots with 9 or fewer crossings have a distinct Jones polynomial. The HOMFLY polynomial, unlike the other two, is multivariable. However, it does maintain a similar structure to that of the Alexander polynomial, using L_+, L_-, and L_0. Knots under both the HOMFLY and Jones polynomials are not affected by orientation, however, when computing the HOMFLY of a link, orientation between the two links does affect the result.

Consider the rules of the HOMFLY polynomial:

$$P(\bigcirc) = 1 \quad (1)$$

$$\alpha P(L_+) - \alpha^{-1} P(L_-) = z P(L_0) \quad (2)$$

The Alexander and Jones polynomials can be derived from the HOMFLY rules as follows:

$$\Delta(t) = P(\alpha = 1, z = t^{-\frac{1}{2}} - t^{\frac{1}{2}})$$

$$V(t) = P(\alpha = t^{-1}, z = t^{\frac{1}{2}} - t^{-\frac{1}{2}})$$

Conclusion

Each polynomial representation of knots has its own benefits and drawbacks. While the HOMFLY polynomial comes the closest to distinguishing between all knots and links, there is not currently any polynomial representation of knots that can completely distinguish all knots and links. Knots are the best!

References and Acknowledgements

It was fascinating to read and learn about how knots, simple strings in space, can be transformed into different polynomials. We would like to thank our graduate mentor Melody Molander, and the DRP, for creating this space for us to explore and grow our interests in mathematics.

Adams, Colin. The Knot Book. American Mathematical Society, 2004.