Skein Theory for Affine ADE Subfactor Planar Algebras

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Quantum Groups Seminar

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Motivation: Galois Theory

$F \subset K$, fields	**subfactor**: an inclusion $N \subset M$ of von Neumann algebras with trivial centers
degree of the field extension $[K : F]$	**index** of the subfactor $N \subset M$ $[M : N]$
the automorphism group Gal(K/F)	the **standard invariant**, a tensor category
	the **principal graph** describes some data of the standard invariant

The index, standard invariant, and principal graph are all invariants of the subfactor!

Example: How does the principal graph encode data?

Tensor category: Category of finite-dimensional representations of S_3

Finite-dimensional irreducible representations of S_3:

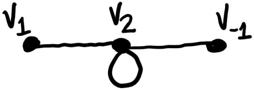
name	denoted by	dimension
trivial	V_1	1
sign	V_{-1}	1
standard	V_2	2

Self-dual object: V_2, i.e., $\overline{V_2} = V_2$

Tensor decompositions:

$$V_1 \otimes V_2 \cong V_2$$
$$V_{-1} \otimes V_2 \cong V_2$$
$$V_2 \otimes V_2 \cong V_1 \oplus V_{-1} \oplus V_2$$

Principal Graph:



Index: $||A||^2$ of $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Eigenvalues of A^*A: 0, 1, 4

$\implies ||A|| = 2 \implies$ index is 4

Index

Theorem (Jones 1983)

Let $N \subset M$ be a subfactor. Then the index $[M : N]$ must lie in the set

$$\{4\cos^2\left(\frac{\pi}{n}\right) | n \geq 3\} \cup [4, \infty]$$

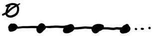
and moreover, all the numbers in this set can be realized as the index of a subfactor.

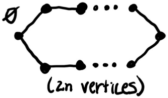
Subfactors are classified up to index $5\frac{1}{4}$.

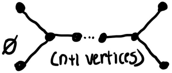
Principal Graphs: A finer invariant of subfactors

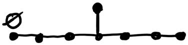
Theorem (Popa 1994)

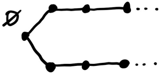
Principal graphs of index 4 subfactors are exactly the simply-laced affine Dynkin diagrams.

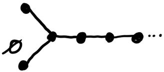
- A_∞ 

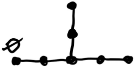
- $\tilde{A}_{2n-1}$ 

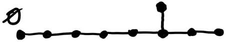
- $\tilde{D}_n$ 

- $\tilde{E}_7$ 

- $\tilde{A}_\infty$ 

- $\tilde{D}_\infty$ 

- $\tilde{E}_6$ 

- $\tilde{E}_8$ 

The Standard Invariant

Theorem (Jones 1999)

Given a finite index subfactor, its standard invariant forms a (shaded) subfactor planar algebra.

Theorem (Popa 1995)

Given a (shaded) subfactor planar algebra $\mathcal{P}$, there is a subfactor whose standard invariant is $\mathcal{P}$.

Goal

The Kuperberg Program

Give a diagrammatic presentation by generators and relations for every subfactor planar algebra.

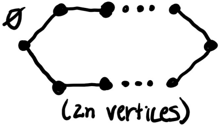
History

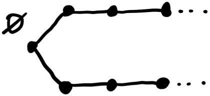
- Index < 4:
  - A_n: (Temperley-Lieb)
  - D_{2n}: Morrison, Peters, Snyder (2008)
  - E_6, E_8: Bigelow (2009)
- Index > 4:
  - A_∞: (Temperley-Lieb)
  - Haagerup & its dual: Peters (2009)
  - Extended Haagerup & its dual: Bigelow, Morrison, Peters, Snyder (2009)
  - 2221 & its complex conjugate: Han (2010)
  - 3311 & its dual: Morrison and Penneys (2013)
  - ...ongoing research...

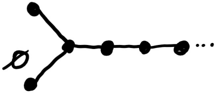
Goal for Today's Talk:

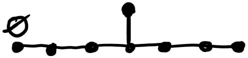
Find presentations for the subfactor planar algebras of index 4 for the:

1. A_∞ Dynkin diagram: 

2. $\tilde{A}_{2n-1}$ Dynkin diagram: 

3. $\tilde{A}_\infty$ Dynkin diagram: 

4. $\tilde{D}_\infty$ Dynkin diagram: 

5. $\tilde{E}_7$ Dynkin diagram: 

But first... What is a planar algebra?

Goal 1: The Temperley-Lieb planar algebra $\mathcal{TL}$

The planar algebra $\mathcal{TL}$ contains the algebras $\mathcal{TL}_k, k \geq 0$, over $\mathbb{C}$.

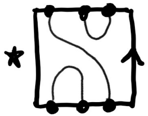
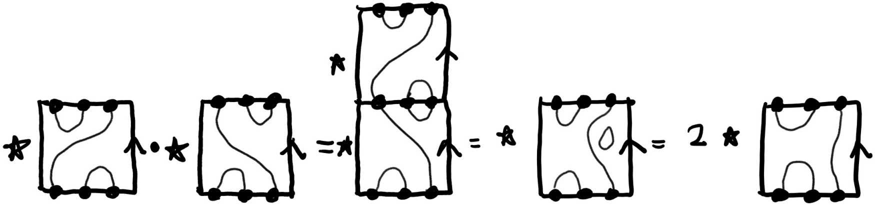


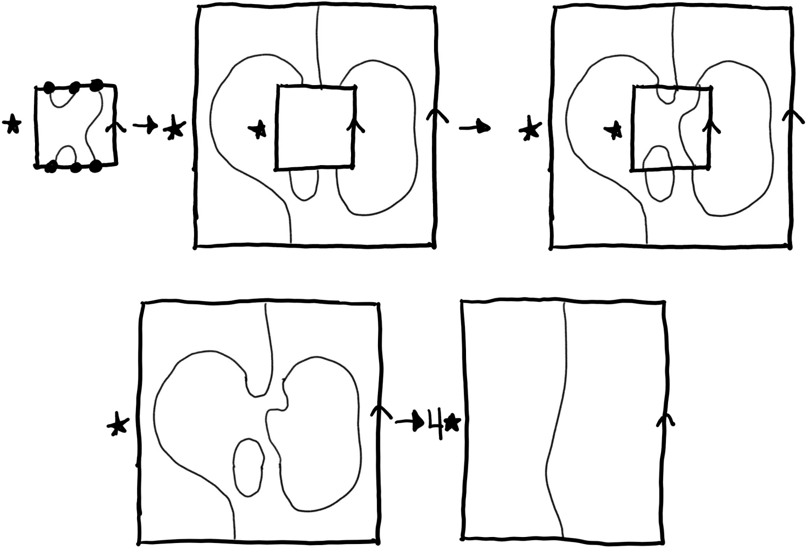
Figure: An element of $\mathcal{TL}_3$

index is 4

$$D \sqcup \bigcirc = 2\,D$$



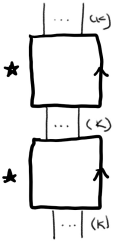
Planar Tangles



Common Planar Tangles

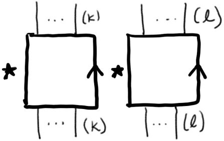
- Multiplication

  $\cdot_k : \mathcal{TL}_k \otimes \mathcal{TL}_k \to \mathcal{TL}_k$



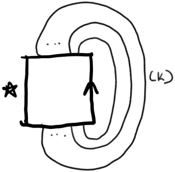
- Tensor

  $\otimes_{k,\ell} : \mathcal{TL}_k \otimes \mathcal{TL}_\ell \to \mathcal{TL}_{k+\ell}$

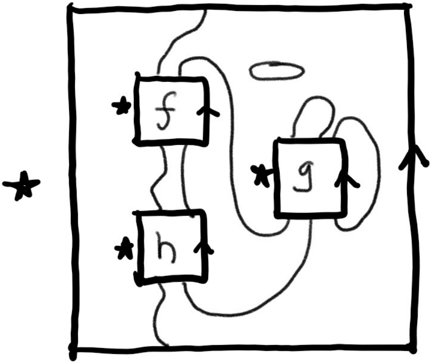


- Trace

  $\mathrm{tr}_k : \mathcal{TL}_k \to \mathcal{TL}_0$



General Planar Tangle



Associated linear map: $Z_T : \mathcal{TL}_2 \otimes \mathcal{TL}_3 \otimes \mathcal{TL}_2 \to \mathcal{TL}_1$

Generalizing $\mathcal{TL}$

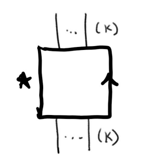
Theorem (Jones)

Any subfactor planar algebra of index $n \geq 4$ contains a copy of $\mathcal{TL}$.

The Formal Definition of a Planar Algebra

Definition

A **planar algebra** is:

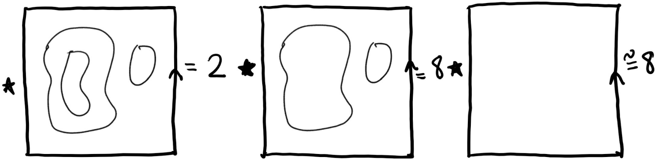
- a collection of vector spaces: $\mathcal{P}_k$, $k \geq 0$
- a rule that assigns to each planar tangle T, a linear map Z_T from the vector spaces associated to the inner squares to the vector space associated to the outer square such that:
  - linear maps compose in the same way planar tangles do
  - isotopy (rel the boundary) give the same maps
  - the identity on $\mathcal{P}_k$ is 

Subfactor Planar Algebra

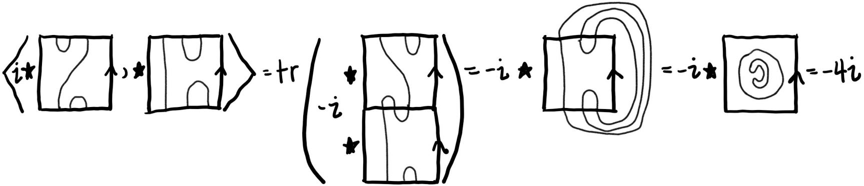
$\mathcal{TL}$ is a special type of planar algebra called a **subfactor planar algebra**, one of the most important properties being:

- $\mathcal{TL}_0$ must be one-dimensional

Why is this a great property?
Any closed diagram evaluates:



and also this gives a that trace will be a sesquilinear form on $\mathcal{P}_k$:

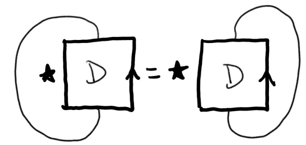


The Formal Definition of a Subfactor Planar Algebras

Definition

A **subfactor planar algebra** is a planar algebra with

- $\mathcal{P}_0$ being one-dimensional
- only spaces for discs with an even number of boundary points are nonzero
- the spherical property, i.e., for each $D \in \mathcal{P}_1$,



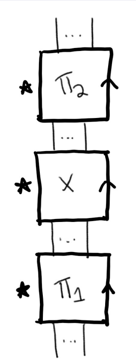
- an antilinear adjoint operation $* : \mathcal{P}_k \to \mathcal{P}_k$ such that the sesquilinear form given by $< x, y >= \mathrm{tr}(x^* y)$ is positive definite. Further $*$ should be compatible with the horizontal reflection operation $*$ on planar tangles.

Create a Tensor Category

$\mathcal{C}_\mathcal{P}$

Give a planar algebra $\mathcal{P}$ we can construct a tensor category $\mathcal{C}_\mathcal{P}$ as follows:

- An object is a projection in one of the n-box spaces $\mathcal{P}_n$ (i.e. $\pi \in \mathcal{P}_n$ such that $\pi^2 = \pi$ and $\pi^* = \pi$)
- Given two projections $\pi_1 \in \mathcal{P}_n$ and $\pi_2 \in \mathcal{P}_m$, define $\mathrm{Hom}(\pi_1, \pi_2)$ to be the space $\pi_2 \mathcal{P}_{n \to m} \pi_1$, i.e, diagrams like:



- The tensor product $\pi_1 \otimes \pi_2$ agrees with $\otimes$ in the planar algebra (placing diagrams side-by-side)
- The dual $\overline{\pi}$ of a projection is rotating it 180 degrees
- The trivial object $\emptyset$ is the empty picture (which is a projection in $\mathcal{P}_0$)

There is a special self-dual object $|$, a single strand.

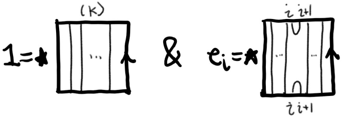
Create a Matrix Category

Mat($\mathcal{C}_\mathcal{P}$)

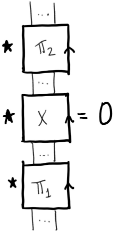
Given the category $\mathcal{C}_\mathcal{P}$ we can define its matrix category:

- The objects of Mat($\mathcal{C}_\mathcal{P}$) are formal direct sums of objects of $\mathcal{C}_\mathcal{P}$ (which were the projections from all the box spaces)
- A morphism of Mat($\mathcal{C}_\mathcal{P}$) from $A_1 \oplus \ldots \oplus A_n \to B_1 \oplus \ldots \oplus B_m$ is an m-by-n matrix whose (i, j)th entry is in $\mathrm{Hom}_{\mathcal{C}_\mathcal{P}}(A_j, B_i)$.
- An induced tensor product from $\mathcal{C}_\mathcal{P}$:
    - Tensoring Objects: formally distribute, i.e., $(\pi_1 \oplus \pi_2) \otimes \pi_3 = (\pi_1 \otimes \pi_3) \oplus (\pi_2 \otimes \pi_3)$
    - Tensoring Morphisms: use usual tensor product of matrices and the tensor product for $\mathcal{C}_\mathcal{P}$ on matrix entries

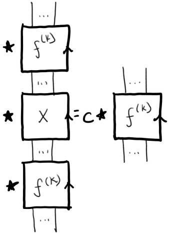
- The **Jones-Wenzl projection** $f^{(k)} \in \mathcal{TL}_k$ is the unique projection in $\mathcal{TL}_k$ with the property that $f^{(k)} e_i = e_i f^{(k)} = 0, \forall i$
  $\mathcal{TL}_k$ generators:



- $\mathcal{TL}$ is *semisimple*: every projection is a direct sum of minimal projections and for any pair of non-isomorphic minimal projections π_1 and π_2 we have that $\text{Hom}(\pi_1, \pi_2) = 0$



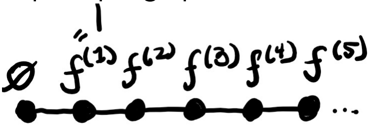
- $f^{(k)}$ are actually *minimal* projections: $\text{Hom}(f^{(k)}, f^{(k)})$ is 1-dimensional:



- Two projections, π_1, π_2, are *isomorphic* if there exists $g \in \text{Hom}(\pi_1, \pi_2)$, and $g^* \in \text{Hom}(\pi_2, \pi_1)$ such that $gg^* = \pi_2$ and $g^* g = \pi_1$.

Principal Graphs

- Principal graphs encode data of $\mathrm{Mat}(\mathcal{C}_P)$
- For $\mathcal{TL}$, **Wenzl's relation**: $f^{(k)} \otimes | \cong f^{(k+1)} \oplus f^{(k-1)}$, $f^{(j)} \in \mathcal{TL}_j$ are the Jones-Wenzl projections
- A principal graph encodes this relationship of the minimal projections:



which is A_∞!

- The **principal graph** of a semisimple planar algebra has vertices the isomorphism classes of minimal projections, and there are

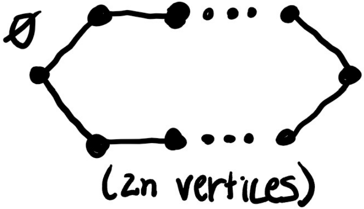
$$\dim \mathrm{Hom}(\pi_1 \otimes |, \pi_2)(= \dim \mathrm{Hom}(\pi_1, \pi_2 \otimes |))$$

edges between the vertices $\pi_1 \in \mathcal{P}_n$ and $\pi_2 \in \mathcal{P}_m$.

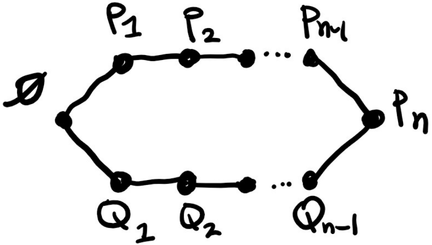
- i.e., Let π be a minimal projection. Then $\pi \otimes | \cong \oplus(\text{neighbors of } \pi \text{ in the principal graph})$

Goal 2

Find the subfactor planar algebras of index 4 associated with the $\tilde{A}_{2n-1}$ Dynkin diagram:



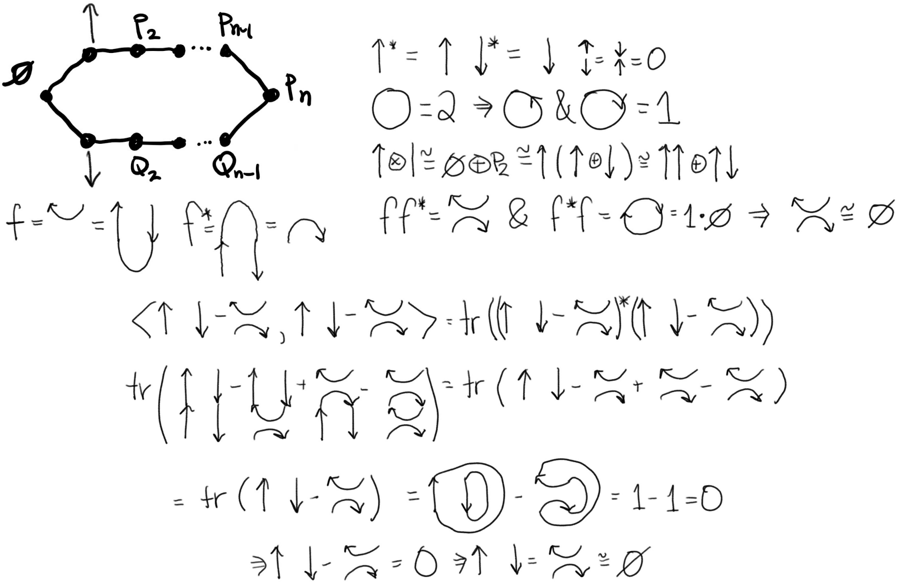
What must happen?



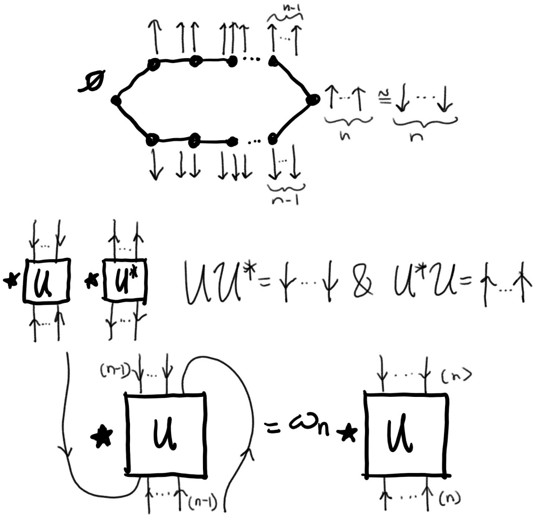
$$| \cong \emptyset \otimes | \cong P_1 \oplus Q_1$$

$$| = \uparrow \oplus \downarrow \quad \text{or} \quad | = | \oplus |$$

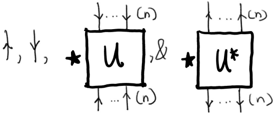
Arrow Case: P_2 and Q_2



Arrow Case: The rest of the vertices

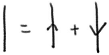


Fix n. Let ω_n be a $2n$th root of unity. Let $\mathcal{P}(U)$ be the planar algebra with

generators: 

and relations:

1. 

2. 
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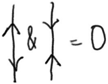
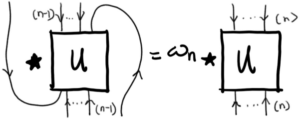
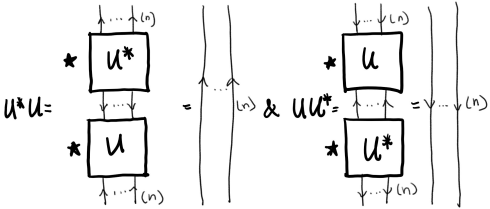
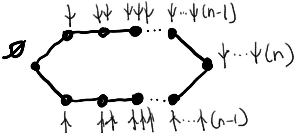
4. 

5. 

6. 

Then this is an $\tilde{A}_{2n-1}$ subfactor planar algebra of index 4 with principal graph:



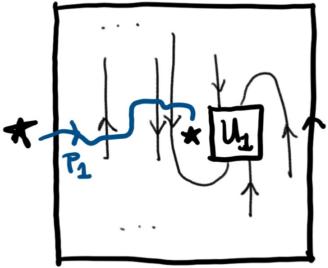
$\mathcal{P}_0(U)$ is at least one-dimensional

Define $f : \mathcal{P}_0(U) \to \mathbb{C}$ by the following algorithm.

1. Define $f(\emptyset) = 1$. Let $D \in \mathcal{P}_0(U)$ be non-Temperley-Lieb. Use the relation (2) to ensure every strand is oriented.
2. Enumerate all the U and U^*. Say $\{U_1, \ldots, U_\ell\}$ and $\{U_1^*, \ldots, U_m^*\}$
3. For each U_i in $\mathcal{P}_0(U)$ make a path, p_i from the star on the outside of the diagram to the star for the U_i. Do similarly for U_i^*.
4. Define the star path length,

   $k_i, k_i^* = (\#\leftarrow) - (\#\rightarrow)$ crossed

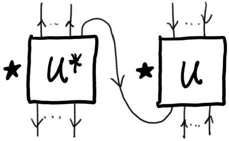
   for each U_i or U_i^*, respectively
5. Calculate $k = \sum_{i,j}(k_i - k_j^*) \mathrm{mod}(2n)$ Then define $f(D) = \omega_n^k$.

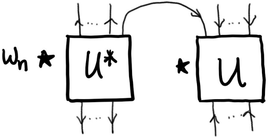


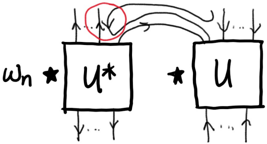
$\mathcal{P}_0(U)$ is at most one-dimensional

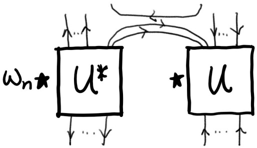
Evaluation Algorithm:

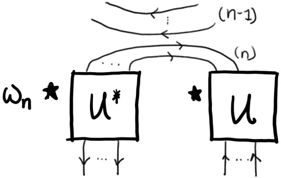
1. Consider a connected component of your diagram. Assume you have no closed loops.

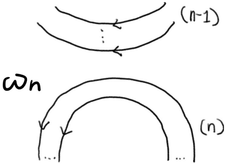
2. 

3. 

4. 

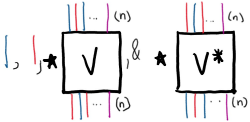
5. 

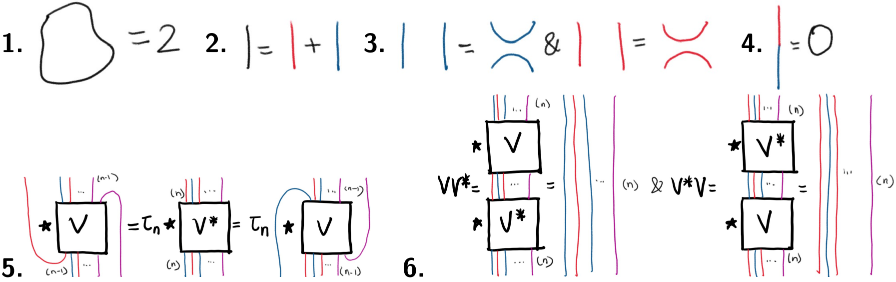
6. 

7. 

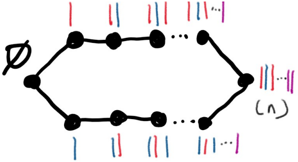
Theorem 2 (M.)

Fix n. Let τ_n be an nth root of unity. Let $\mathcal{P}(V)$ be the planar algebra with

generators:  and relations:



Then this is an $\tilde{A}_{2n-1}$ subfactor planar algebra of index 4 with principal graph:



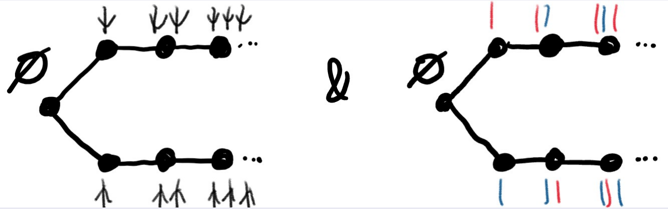
Goal 3

Proposition 1

There are at exactly $3n$ distinct subfactor planar algebras with principal graphs $\tilde{A}_{2n-1}$.

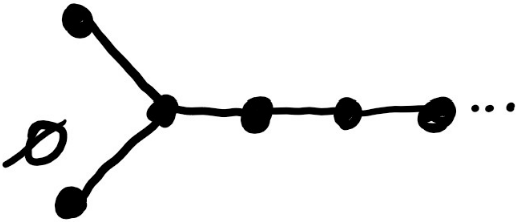
Proposition 2

There are exactly two distinct subfactor planar algebras with principal graphs $\tilde{A}_{\infty}$, i.e, the principal graphs are:

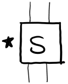


Goal 4

Find the subfactor planar algebra(s) of index 4 associated with the $\tilde{D}_\infty$ Dynkin diagram:

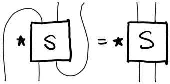


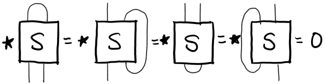
Theorem 3 (M.)

Let $\mathcal{P}(S_2)$ be the planar algebra with generator: 

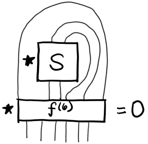
and relations:

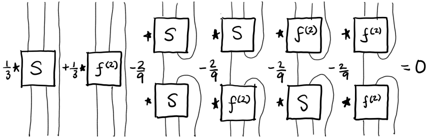
1. 

2. 

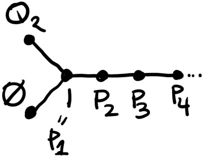
3. 

4. $S^2 = S + 2f^{(2)}$

5. 

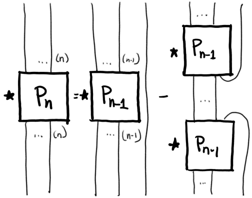
6. 

Then $\mathcal{P}(S_2)$ is a $\tilde{D}_\infty$ subfactor planar algebra of index 4 with principal graph:



Further, any $\tilde{D}_\infty$ subfactor planar algebra has this presentation.

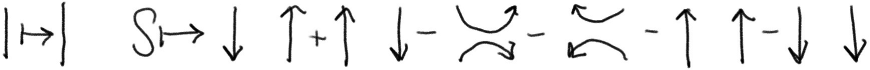
where $Q_2 = \frac{1}{3}S + \frac{1}{3}f^{(2)}$,
$P_2 = -\frac{1}{3}S + \frac{2}{3}f^{(2)}$, and



$\mathcal{P}(S_2)_0$ is at least one-dimensional

Proposition 3

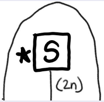
The $\tilde{D}_\infty$ subfactor planar algebra is a subplanar algebra of the arrow case of the $\tilde{A}_\infty$ planar algebra.

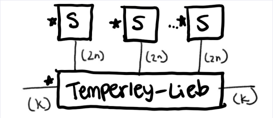


Evaluation Algorithm

Theorem (Bigelow and Penneys 2012)

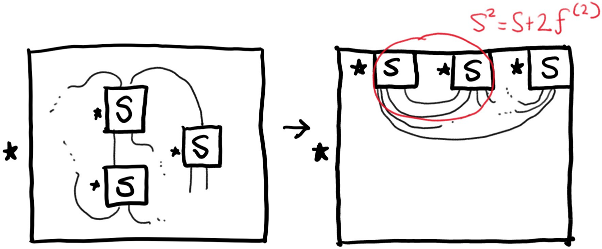
If a planar algebra generated by $S \in \mathcal{P}_n$ satisfies that:

- 

is a linear combination of trains, i.e., 

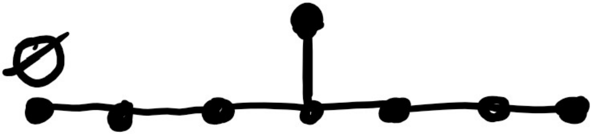
,
- and S^2 is a linear combination of S and $f^{(n)}$

then any closed diagram can be evaluated using the **jellyfish algorithm.**

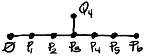


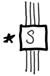
Goal 5

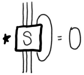
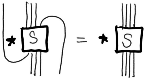
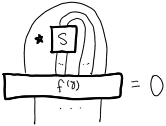
Find the subfactor planar algebra(s) of index 4 associated with the $\tilde{E}_7$ Dynkin diagram:

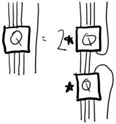


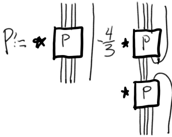
Theorem 4 (M.)

Let $\mathcal{P}$ be a subfactor planar algebra with principal graph $\tilde{E}_7$: 

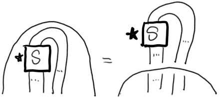
then $\mathcal{P}$ is generated by  with relations:

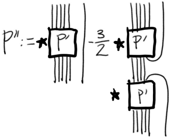
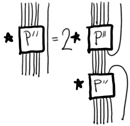
1.  **2.**  **3.**  **4.** $S^2 = S + 6f^{(4)}$ **5.** 

6. For $Q = \frac{2}{5}f^{(4)} + \frac{1}{5}S$, 

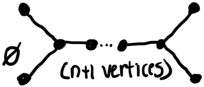
7. For $P = \frac{3}{5}f^{(4)} - \frac{1}{5}S$,  &

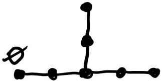
8. Defining ,
satisfies Reidemeister 2 & 3, &

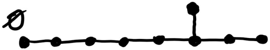


 then 

Future Work

1. Prove the other direction of $\tilde{E}_7$.
2. Find similar results for the other affine Dynkin diagrams:
    - $\tilde{D}_n$ 

    - $\tilde{E}_6$ 

    - $\tilde{E}_8$ 

Thank you!

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