Melody Molander's Research Statement

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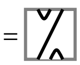
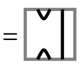
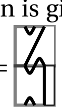
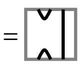
Overview

My interests are in *quantum algebra* and *quantum topology*, which lie at the intersection of many fields of mathematics. I use tools primarily from higher category theory, knot theory, and subfactor theory to research *quantum symmetry*. Classical objects such as polygons and vector spaces are highly symmetric, and the language of groups helps describe these symmetries. However, objects from quantum mechanics have more complex symmetries that can no longer be captured through group theory. These *quantum symmetries* instead require the language of *2-categories*. A 2-category is a higher category with not only objects and morphisms, but also *2-morphisms* between morphisms. These 2-categories are advantageous because they have a diagrammatic description which allows the use of topology, akin to knot theory. Just as groups are ubiquitous in mathematics, 2-categories are seen in a variety of subjects such as operator algebras, representation theory, topology, and mathematical physics.

Classification of Subfactors and C*-Algebras

An operator algebraic object called a *subfactor* leads to a rich source of interesting quantum symmetries. The *standard invariant* of a subfactor is a 2-category encoding the quantum symmetry. I research quantum symmetry by studying subfactors, their standard invariants, and topics that stem from these concepts.

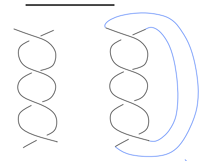
A *planar algebra*, introduced by Jones [25], is a diagrammatic way to view the standard invariant of a subfactor. 2-morphisms can be drawn as pictures in the plane and multiplication is given by vertical stacking. That is, if 2-morphisms f and g can be drawn as $f =$  and $g =$  then $fg =$  $=$  $= g$. Since the pictures satisfy that $fg = g$, this must be a true equality in the standard invariant. Picture computations are simple, so knowing the generators (what, beyond black strands, make up pictures of 2-morphisms) along with relations of planar algebras is useful for exploring quantum symmetry.

Subfactors have another invariant, a real number called the *index*. Jones [23] found that the indices of subfactors are given by the set $\{4\cos^2\left(\frac{\pi}{n}\right) | n \geq 3\} \cup [4, \infty]$. Decades of work have been done to classify subfactors of small index. Planar algebras have frequently been used to construct and classify subfactors [3, 5, 36, 42].

Many major breakthroughs in classifying a different operator algebraic object called a *C*-algebra* have strong parallels to the subfactor theory framework. For a C*-algebra to be *classifiable*, it must be simple and have finite *nuclear dimension*, among other criteria. Expanding classification to the nonsimple setting has been of high interest in recent years (see [6, 14, 16]). In this setting, nuclear dimension remains a pivotal invariant and determining its value is a problem of interest.

Topological Quantum Computing

Figure 1



Jones' [24] research in subfactors led to a knot invariant called the *Jones polynomial*. A *knot* can be thought of as a knotted string with its ends glued together. Determining if two knots are distinct up to isotopy is incredibly difficult. The Jones polynomial takes in a knot and produces a polynomial. If two knots produce different polynomials, they must be distinct knots.

While the Jones polynomial is powerful for distinguishing knots, it is also $\#P$-hard [21]. There are physical systems which can compute the Jones polynomial called *TQFT*s. Freedman, Kitaev, Larsen, and Wang [19] showed that a topological quantum computer can efficiently simulate TQFTs and in turn will approximate the Jones polynomial. Kitaev's [29] proposal of topological quantum computing is to build a computer that operates by manipulating motions of particle-like objects called *anyons*. The trajectories of anyons form braids (Figure 1, left) whose closures form knots (Figure 1, right) or collections of knots, called *links*.

Anyons can be represented as objects in *fusion categories* which can arise from particular subfactors. Fusion rules in these categories encapsulate the physics of anyons. That is, a rule in the category such as $X \otimes Y = Z \oplus W$ tells us that anyons X and Y fuse to become the anyon Z half the time and the anyon W the other half of the time. These rules appear through diagrammatic relations in the planar algebras. Thus, planar algebras are also of active interest for their applications to topological quantum computing.

My Results

In my research thus far I have:

1. Constructed generators-and-relations descriptions of planar algebras for index 4 subfactors [33, 34];
2. Described diagrammatics of fusion categories with affine ADE fusion rules [33, 34];
3. Proved equivalence of categories arising from subfactors to other well-known categories [33];
4. Bounded the nuclear dimension of a particular class of C*-algebras [18];
5. Found classes of knots whose Jones polynomials can be computed in polynomial time on a classical computer [2].

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Goals of My Research Program

- Complete giving diagrammatic descriptions of all index 4 planar algebras.
- Develop diagrammatics to accommodate the extra fusion categories of $\mathrm{Vec}^{\omega}_{D_{2n}}$ that do not appear as planar algebras.
- Describe diagrammatically the representation category for the binary octahedral group.
- Create new categories using commutative algebra objects of the affine A finite category.
- Bound nuclear dimension of Cuntz-Pimsner algebras arising from line bundles and nonperiodic homeomorphisms.
- Find classes of knots for which the colored Jones polynomials can be computed in polynomial time on a classical computer.

The rest of this proposal will be devoted to addressing the results on the first page as well as discussing in detail the goals of my research program.

1. Using Diagrams to Understand the Classification of Index 4 Subfactors

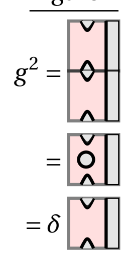
A *von Neumann algebra* is a unital algebra of bounded operators on a Hilbert space closed under a specific topology and possessing an *adjoint* operation that can be thought of as a generalized matrix transpose. When the Hilbert space is finite-dimensional, von Neumann algebras are necessarily direct sums of the algebras of $n \times n$ matrices over $\mathbb{C}$ (with varying n). *Factors*, which are von Neumann algebras with center isomorphic to $\mathbb{C}$, are building blocks of von Neumann algebras. A unital inclusion of factors, $N \subseteq M$, is called a *subfactor*.

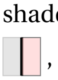
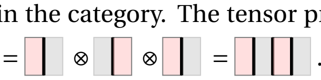
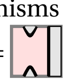
Subfactor theory can be thought of as a noncommutative version of Galois theory. While Galois theory studies inclusions of fields $F \subseteq E$, subfactors are unital inclusions, $N \subseteq M$, of factors, which are highly noncommutative algebras. Both fields and factors satisfy that any map between them are unital inclusion or zero, so analyzing maps between fields or factors is the same as studying field extensions or subfactors. The index $[M : N]$ of a subfactor measures the size of these extensions, analogous to the degree $[E : F]$ of a field extension.

The standard invariant 2-category is analogous to the Galois group. Let $\mathscr{R}$ be the *hyperfinite II_1 factor* and G be a finite group. There is a unique way, up to conjugacy, in which G can act on $\mathscr{R}$ by outer automorphisms [22, 12]. Let $N = \mathscr{R}^G$, the fixed points of the action. N can be shown to be a factor. There will be a Galois correspondence between intermediary subfactors $N \subseteq M \subseteq \mathscr{R}$ and subgroups $H \leq G$ [39, 40]. Just like in Galois theory, the automorphisms fixing N of M, $\mathrm{Aut}_N(M)$, equals G. Additionally, $|G| = [\mathscr{R} : N]$. However, not all examples of finite index subfactors come from groups.

To research subfactors, we look at their standard invariant 2-categories. These categories describe quantum symmetry. One way to study classical symmetry is to consider how a finite group, G, acts on a vector space through the category of modules over the group, $\mathrm{Rep}(G)$. Quantum symmetry is then the noncommutative analogue of the representation category. For a subfactor, $N \subseteq M$, we examine the bimodules coming from N and M to build the standard invariant and understand quantum symmetry.

Figure 2



The objects of the 2-category are M and N. Diagrammatically, N is "pink-shaded" and M is "gray-shaded". There are two special bimodules, an $N - M$ bimodule X, , and an $M - N$ bimodule $\overline{X}$, , which are morphisms in the category. The tensor product is represented by placing diagrams side-by-side, e.g., $X \otimes \overline{X} \otimes X =$ . All other morphisms arise as a summand of tensor products of X and $\overline{X}$. The 2-morphisms are bimodule intertwiners. An example of a 2-morphism between $X \otimes \overline{X} \otimes X$ and itself is: $g =$  since the top and bottom boundary of g are the same as $X \otimes \overline{X} \otimes X$ (alternating pink-gray-pink-gray between 3 black strands).

Multiplication of 2-morphisms is given by vertical stacking. Circles evaluate to a real number δ. Figure 2 gives an example of this multiplication. The value of δ^2 is the index of the subfactor. The endomorphism spaces of 1-morphisms along with an action of a structure called a *planar operad* on these spaces form a *planar algebra*.

Kuperberg posed a program to see how far the planar algebra language can be pushed in the understanding of the standard invariants of subfactors.

The Kuperberg Program: *Provide a presentation by generators-and-relations for every subfactor planar algebra and prove as many properties as possible about the planar algebra using only this presentation.*

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As a graduate student, my aim was to make progress on the Kuperberg program for index 4 planar algebras. At index 4, there is a classification of subfactors by Popa [43] showing an invariant of a subfactor called the *principal graph* must be an affine ADE Dynkin diagram. For planar algebras with principal graph affine A finite, affine A_∞, affine D finite, affine D_∞ and affine E_7, I [33, 34] have found generators-and-relations presentations and gave novel proofs of known properties of the planar algebras strictly through these presentations. *As these diagrams take up space, I will only give below the generators of one case and one relation.*

Theorem 1 *(M. [33]) All index 4 planar algebras with principal graph affine A_{2n-1} have generators given in Figure 4 (where the U, V, U^*, and V^* boxes have n strands on the top and bottom, alternating in color). The relations are governed by an nth root of unity, ζ_n. There are n nonisomorphic planar algebras of this type. One relation is given in Figure 3.*
(Similar presentations are found for affine A_∞, affine D finite, affine D_∞, and affine E_7 [33, 34].)

Figure 3

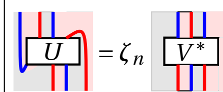
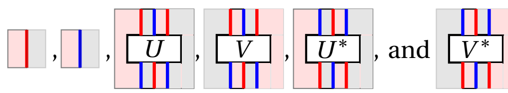
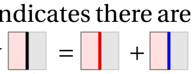


Figure 4



This theorem helps us better visualize the standard invariants of index 4 subfactors. By showing how many nonisomorphic planar algebras of this type exist, this theorem also serves as a new proof of a subfactor classification result by Popa [43] using the planar algebra language.

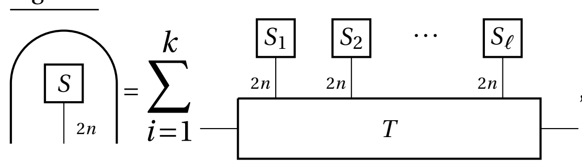
To identify the generators-and-relations in Theorem 1, I utilized data from the principal graph. For example, if the graph indicates there are elements satisfying $X = P_1 \oplus Q_1$, I define corresponding elements in the planar algebra that satisfy  , labeling them X, P_1, and Q_1, respectively. To demonstrate that the generators-and-relations in Theorem 1 are sufficient, I prove the identified relations satisfy all properties required to describe a subfactor's standard invariant. The most challenging property to establish is that the endomorphism spaces of diagrams with zero strands on the top and bottom boundary are one-dimensional. To show this, I define a *jellyfish algorithm*, which maps from an endomorphism space of a planar algebra to $\mathbb{C}$. If the function is well-defined and surjective, it follows that the space of diagrams with zero strands is at least one-dimensional. Then I show the space is at most one-dimensional by defining an evaluation algorithm on these diagrams.

Future Work in Index 4 Planar Algebras

Problem 1: *Complete the Kuperberg Program for index 4 planar algebras. That is, construct diagrammatic descriptions for the remaining index 4 planar algebras: those with principal graph affine E_6 and affine E_8.*

The jellyfish algorithm involves identifying generators that satisfy two specific *skein relations* (shown in Figure 5). I have already found such generators for affine E_8, which closely resemble those for affine E_7, except with more strands. This suggests that the presentations and proofs for affine E_8 will be similarly structured. In contrast, affine E_6 presents a challenge, though it remains approachable since the principal graph is *spoke*. Works of Bigelow, Morrison, Penneys, and Peters [4, 35, 41] have explored finding generators of planar algebras with spoke principal graphs, indicating that finding generators that satisfy the algorithm's needs is promising.

Figure 5

(i) 

where T is some diagram with no S, and S is a generator with $2n$ strands and the appropriate shadings

(ii) S^2 is a linear combination of diagrams with at least one less S.

2. Index 4 Planar Algebras Arising from Fusion Categories

By dropping the "pink" and "gray" shading, we can consider *unshaded planar algebras*. These are defined to satisfy the same axioms as shaded planar algebras, except they no longer represent the standard invariant of a subfactor. Henceforth, planar algebras will be assumed to be unshaded.

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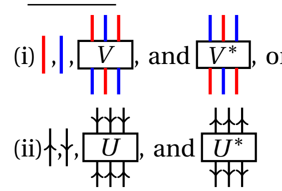
These planar algebras are still worth pursuing. The corresponding categories are *fusion categories*, as long as the principal graph is finite. Every fusion category is the representation category of a *weak Hopf algebra* [45], so these categories still capture quantum symmetry. Furthermore, these fusion categories describe anyonic systems relevant to topological quantum computing.

I have given presentations of affine ADE planar algebras [33, 34] in a similar process to Theorem 1. One theorem obtained *(reduced to just listing the generators for spacing reasons)* is:

> ***Theorem 2*** *(M. [33]) There are two types of index 4 subfactor planar algebra with principal graph affine A_m, subject to some relations governed by a root of unity. These two types of generators are given in Figure 6 (where the number of strands on the boxes depend on m).*
> *Case (i) does not exist when m is even and there are $(m+1)/2$ nonisomorphic planar algebras when m is odd which depend on an $(m+1)/2$th root of unity.*
> *Case (ii) always has $m+1$ nonisomorphic planar algebras depending on an $(m+1)$th root of unity.*
> *(Similar presentations are found for affine A_∞, affine D finite, affine D_∞, and affine E_7 [33, 34].)*

Figure 6



To better understand quantum symmetry and contribute to the Kuperberg program, we should explore relationships between categories that can be proven through the planar algebra language. One of the fundamental fusion categories is the category of finite-dimensional vector spaces, graded by a finite group G with *associator* described by a 3-cocycle ω, Vec_G^ω. When $G = \mathbb{Z}_{m+1}$, each Vec_G^ω fusion category is governed by an $m+1$ root of unity, ζ [13]. These ζ and ω are in bijective correspondence, so let's rename our category, $\mathrm{Vec}_{\mathbb{Z}_{m+1}}^\zeta$. My results in Theorem 2 then give diagrammatics of these categories:

> ***Theorem 3*** *(M. [33]) Let $\mathscr{P}_\zeta$ be an affine A_m planar algebra of type (ii) from Theorem 2 with root of unity ζ. As a fusion category, $\mathscr{P}_\zeta$ is monoidally equivalent to $\mathrm{Vec}_{\mathbb{Z}_{m+1}}^{\zeta^{-1}}$. Further, we can diagrammatically represent <u>all</u> $\mathrm{Vec}_{\mathbb{Z}_{m+1}}^\omega$ categories as a $\mathscr{P}_\zeta$ category for some root of unity ζ.*

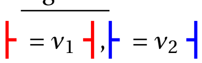
The fusion categories $\mathrm{Vec}_{\mathbb{Z}_m}^\omega$ are not strict when the associator ω is not the identity. That is, we can only have an isomorphism, not an equality, in the following statement: for all objects X, Y, Z in the category, $X \otimes (Y \otimes Z) \cong (X \otimes Y) \otimes Z$. However, every monoidal category is monoidally equivalent to a strict monoidal category [31]. From the planar algebra presentations we construct in Thoerem 2, I show the equivalence of the nonstrict $\mathrm{Vec}_{\mathbb{Z}_m}^\omega$ categories to the strict categories coming from the planar algebras explicitly. As a result, this shows how the associator gets hidden in the skein theory of the planar algebras.

Future Work in Fusion Categories from Index 4 Planar Algebras

Theorem 3 raises the question: which fusion categories are represented by case (i) of Theorem 2? These are the fusion categories $\mathrm{Vec}_{D_{2n}}^\omega$, where $m = 2n-1$. However, there are only n nonisomorphic planar algebras of this type, whereas there are $4n$ or $2n$ nonisomorphic fusion categories $\mathrm{Vec}_{D_{2n}}^\omega$, depending on if n is even or odd. The planar algebra framework of case (i) does not encompass all of these fusion categories.

> ***Problem 2:*** *Develop diagrammatics to describe the $\mathrm{Vec}_{D_{2n}}^\omega$ not described by planar algebra diagrammatics.*

Figure 7



I plan to show Problem 2 by considering the *Frobenius-Schur indicators* of the building blocks of the category (*simple objects*). $\mathrm{Vec}_{D_{2n}}^\omega$ can be endowed with a unique structure where these objects have a *Frobenius-Schur indicator*, ν, with $\nu \in \{0, \pm 1\}$ [15].

There are two self-adjoint simple objects whose identity morphisms are the first two generators in Figure 6(i). Their Frobenius-Schur indicators will be either ± 1. Diagrammatically, this indicator shows what happens when you rotate the morphism representing the identity of that object by π. I conjecture that we can obtain the rest of the presentations by introducing a *tag* on the strands with the relations as shown in Figure 7 (where ν depends on the parity of n). Other skein relations from Theorem 2 will need to be modified, but once completed, this should yield $2n$ or $4n$ distinct presentations, as desired.

3. Equivalence of Index 4 Planar Algebras to Representation Categories

McKay [32] established a one-to-one correspondence between the finite subgroups of $SU(2)$ and the affine ADE Dynkin diagrams in 1980. The rules following from a *representation graph* for a representation category are determined in the same way as the rules following from a principal graph for a category corresponding to a planar algebra. If the representation graph is affine A finite, the associated category is $\mathrm{Rep}(C_m)$, where C_m is a cyclic subgroup of $SU(2)$. Therefore, the planar algebra presentations from Theorem 2 produce another diagrammatic presentation of this category. I provide a new proof of the monoidal equivalence of these categories strictly diagrammatically:

Theorem 4 *(M. [33]) Let $\mathscr{P}_1$ be an affine A_m planar algebra with the root of unity set to 1. As a fusion category, $\mathscr{P}_1$ is monoidally equivalent to Rep(C_m) where C_m is a finite cyclic subgroup of $SU(2)$.*

Future Work in Representation Categories from Index 4 Planar Algebras

To establish Theorem 4, I constructed a functor that provides an equivalence of categories from known diagrammatics of $\mathrm{Rep}(C_m)$ given by Reynolds [44] to $\mathscr{P}_1$. For affine E_7, I have diagrammatics which McKay's result says describe the Rep$(2O)$, where $2O$ is the binary octahedral group. To prove this using my presentation, I need similar diagrammatics given by Reynolds for $\mathrm{Rep}(C_m)$, which do not exist. Beginning with the planar algebra framework, I formulate the following problem.

Problem 3: *Create new diagrammatics of the representation category of the binary octahedral group, Rep$(2O)$, using the planar algebra framework.*

It is challenging to know if a presentation has enough relations. Since $\mathrm{Rep}(C_m)$ and $\mathscr{P}_1$ had essentially the same relations, it is promising that the framework found for affine E_7 will be enough to determine all of the relations of Rep$(2O)$.

4. Relationships Between Nonisomorphic Index 4 Planar Algebras

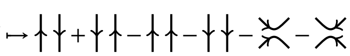
To bound the dimensions of morphism spaces in affine D_∞, I demonstrated that these spaces can be viewed in the affine A_∞ planar algebra, as stated in the following theorem.

Theorem 5 *(M. [34]) The affine D_∞ planar algebra is a subplanar algebra of an affine A_∞ planar algebra.*

The proof of this theorem involves embedding the affine D_∞ planar algebra into the arrow case of the affine A_∞ planar algebra, as depicted in Figure 8. We can also view this theorem as giving that affine D_∞ is a quotient of an affine A_∞ planar algebra. This leads to questions about further relationships between index 4 planar algebras.

Figure 8

For the generator S of the affine D_∞ planar algebra, define the embedding by:

$S \mapsto$ 

Future Work in Relationships of Index 4 Planar Algebras

Since the affine D_∞ case is a subplanar algebra of an affine A_∞ case, it is plausible to conjecture that the affine D finite case is a subplanar algebra of an affine A finite case. By leveraging *commutative algebra objects*, it is possible to form quotients of fusion categories, thereby creating new fusion categories through a process known as *anyon condensation*.

In [28], the authors use this idea to turn commutative algebra objects of A_n planar algebras to presentations of D_{2n} planar algebras. Applying these techniques, I will approach the following problem.

Problem 4: *Create quotients of the affine A finite category to obtain novel presentations of categories such as the affine D finite category.*

5. Classification of C*-Algebras and Nuclear Dimension

C-algebras*, similar to von Neumann algebras, are algebras of bounded operators on a Hilbert space, equipped with an adjoint operation, and closed under a specific topology.

It is expected that the subfactor theory framework can be used to classify C*-algebras. For a subfactor $N \subseteq M$ with a finite principal graph, the standard invariant can also be expressed as a pair $(\mathscr{C}, A)$ where $\mathscr{C}$ is a fusion category of $N-N$ bimodules acting on N, and $\mathscr{A}$ is an *algebra object* representing M and generating $\mathscr{C}$ [37]. In order to classify the subfactor, we need that N is *amenable*. Classification of subfactors then involves understanding the actions of these fusion categories on factors.

Fusion categories actions on, instead, C*-algebras is a topic of great current interest (explored in 2023 and 2024 in work such as [9, 17, 30]). Analogous to N needing to be amenable, we need the C*-algebra *classifiable*, which requires many criteria including that the C*-algebra is simple and of finite *nuclear dimension*.

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Nuclear dimension is a notion of dimension on C*-algebras introduced by Winter and Zacharias [47] as a noncommutative generalization of covering dimension for a topological space. That is, when X is a compact metrizable space, $\dim_{\text{nuc}}(C(X)) = \dim_{\text{cov}}(X)$, where $C(X)$ is the C*-algebra of continuous functions on X. Paramount discovery by Castillejos, Evington, Tikuisis, White, and Winter in 2020 and 2021 [7, 8] resolved the nuclear dimension of all simple C*-algebras. Specifically, they demonstrated that simple C*-algebras have nuclear dimension 0, 1 or ∞.

I am exploring the nuclear dimension of nonsimple *Cuntz-Pimsner algebras of C*-correspondences*, which are thought of as a generalization of the C*-algebra crossed product. Let X be a compact metric space. A *right Hilbert $C(X)$-module*, $\mathscr{E}$, is a right $C(X)$-module equipped with a $C(X)$-valued inner product, that satisfies some natural criteria, making $\mathscr{E}$ complete in the induced norm. A *C*-correspondence* over $C(X)$ is a right Hilbert $C(X)$-module, $\mathscr{E}$, equipped with a map, φ, from $C(X)$ into the space of adjointable operators of $\mathscr{E}$. The *Cuntz-Pimsner algebra of* $\mathscr{E}$, $\mathscr{O}(\mathscr{E})$, is the C*-algebra generated by the *universal covariant representation* of $\mathscr{E}$.

For example, let $\mathscr{V}$ be a vector bundle over X and α be a homeomorphism from X to itself. Then the continuous sections of the vector bundle, $\Gamma(\mathscr{V})$, is a right $C(X)$-module, which admits a right $C(X)$-valued inner product, making $\Gamma(\mathscr{V})$ a right Hilbert $C(X)$-module. We then get a C*-correspondence, $\mathscr{E} = \Gamma(\mathscr{V}, \alpha)$, by defining the map, φ, to be $\varphi(f)(\xi) = \xi f \circ \alpha$. When $\mathscr{V}$ is a trivial line bundle, then the Cuntz-Pimsner algebra of $\mathscr{E}$, $\mathscr{O}(\mathscr{E})$, is isomorphic to the crossed product $C(X) \rtimes_\alpha \mathbb{Z}$.

In 2024 [1] it was found that $\mathscr{O}(\Gamma(\mathscr{V}, \alpha))$ has nuclear dimension at most 1 when X is an infinite compact metric space with finite covering dimension and α is a *minimal* homeomorphism. I am interested in bounds on nuclear dimension when dropping the minimal criteria on the previous result. To that end, with my collaborators: Marzieh Forough, Zahra Hassanpour-Yakhdani, Ja A Jeong, Preeti Luthra, and Karen Strung, we proved that:

Theorem 6 *(M. et al [18]) Let X be a compact Hausdorff second-countable space, α a periodic homeomorphism on X, and $\mathscr{V}$ a line bundle over X. Then $\dim_{nuc}(\mathscr{O}(\Gamma(\mathscr{V}, \alpha))) \leq 2\dim_{cov}(X) + 1$. In particular, $\mathscr{O}(\Gamma(\mathscr{V}, \alpha))$ has finite nuclear dimension if X has finite covering dimension.*

Future Work in Nuclear Dimension

Hirshberg and Wu [20] found that the crossed product $C(X) \rtimes_\alpha \mathbb{Z}$ has finite nuclear dimension when X is a compact, finite-dimensional metric space. Our proof of Theorem 6 relies heavily on the framework they establish, which first requires α is periodic. Building on this result, we aim to further extend Theorem 6 by answering the following problem.

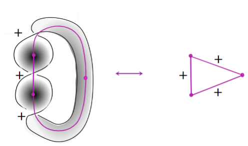
Problem 5: *Without requiring that α is periodic, what is an upper bound of the nuclear dimension for the Cuntz-Pimsner algebra, $\mathscr{O}(\Gamma(\mathscr{V}, \alpha))$, where $\mathscr{V}$ are line bundles over compact Hausdorff second-countable spaces?*

6. The Jones Polynomial and Computational Complexity

A fundamental question of knot theory is determining whether two knots are distinct. A useful tool which has been constructed to answer such a question is the use of knot invariants, such as knot polynomials. In this process, knots are assigned a polynomial, and if the polynomials are distinct for two knots, then it is known that these two knots must be different. Remarkably, Jones' [24] research in subfactors led to a knot invariant called the *Jones polynomial*.

Although knot polynomials are utile, they are very difficult to compute. Thus, knot theorists have sought quicker ways to compute them. A combinatorial approach through the use of graph theory has been of particular interest. One such method consists of assigning a knot a corresponding graph, called a Tait graph. An example of assigning a knot to a Tait graph is shown in Figure 9. Each spanning tree, T, of the Tait graph, Γ, gives a weighting of Γ. Then Thistlethwaite [46] proved that the Jones polynomial can be computed as

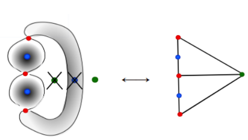
Figure 9



$$J(L) = (-A^{-3})^{wr(L)} \sum_T \prod_{e \in \Gamma} \mu_T(e)$$

where Γ is the signed Tait graph of a link L, T is a spanning tree of Γ, $\mu_T(e)$ is the weight of e associated to a spanning tree T, and $wr(L)$ is the *writhe* of the link.

Figure 10



Cohen, Dasbach, and Russell [10, 11] used Thistlethwaite's work to introduce a similar method that recovers the Jones polynomial for the class of pretzel knots. Specifically, they assign a knot a corresponding graph called a balanced overlaid Tait graph. An example is given in Figure 10. The edges of the graph can be assigned special weightings. Then they prove for pretzel knots that the following summand:

$$(-A^{-3})^{wr(L)} \sum_{P} \prod_{e \in P} \mu_P(e)$$

where P is a perfect matching of the balanced overlaid Tait graph of a link, e is an edge in P, and $\mu_P(e)$ is its weight, gives the same result as the spanning tree method and thus is the Jones polynomial.

The benefit of the perfect matching method is that the authors also found that this sum can be computed by taking a determinant of a specific submatrix of the graph's adjacency matrix. This means that the Jones polynomial for the pretzel knots can be computed in polynomial time on a classical computer.

It is well known that the computational complexity of the Jones polynomial of a knot is $\#P$-hard due to the work of Jaeger, Vertigan, and Welsh [21]. Although it is not possible for all of the Jones polynomials to be computed by this perfect matching method due to the $\#P$-hard restriction, there may exist other examples where the computationally-fast algorithm applies.

I am interested in finding other classes of knots where polynomial time algorithms can be found. To that end, I was a mentor for a Research Experience for Undergraduates (REU) that focused on finding algorithms for classes of knots that produce the Jones polynomial in polynomial time. With Derya Asaner, Sanjay Kumar, Andrew Pease, and Anup Poudel, we adapted Cohen, Dasbach, and Russell's perfect matching method and found:

Theorem 7 *(M. et al [2]) The Jones polynomials for closed braids of the form $\sigma_1^{m_1}\sigma_2^{m_2}...\sigma_n^{m_n}$, where σ_i are the braid generators and exponents are either all negative or positive, can be computed in polynomial time by a perfect matching method on a classical computer.*

In this paper we also provide a polynomial time algorithm to compute another such polynomial knot invariant, *the Kauffman polynomial*, using the matrices obtained from the balanced overlaid Tait graphs of $(2,q)$-*torus knots*, i.e., closed braids of the form σ_1^q, where σ_1 is the braid generator in B_2. We found that:

Theorem 8 *(M. et al [2]) The Kauffman polynomials for $(2,q)$-torus knots can be computed in polynomial time by a perfect matching method on a classical computer.*

Many knot invariants, including the Jones polynomial, are derived from the *HOMFLYPT* polynomial. Theorem 8 is the first known example of generalizing the method of Cohen, Dasbach, and Russell to a knot invariant that cannot be derived from HOMFLYPT.

Future Work in The Jones Polynomial and Computational Complexity

One advantage of having computationally-fast methods are its application in conjectures between the geometry and quantum topology of manifolds. For example, *the volume conjecture*, first posed by Kashaev in [26] and reformulated in terms of the *colored Jones polynomial* by Murakami and Murakami [38], relies on determining asymptotics of the complex calculations involved in the colored Jones polynomial. Therefore, a method in simplifying the calculations may provide useful in such conjectures. This makes me interested in the following problem:

Problem 6: *Find classes of knots for which the colored Jones polynomials can be computed in polynomial time.*

Including the Jones polynomial, many knot invariants can be derived from combinatorial *skein relations* on knot diagrams. Because the proofs of Theorems 7 and 8 relies on the combinatorics of these relations, it is reasonable to believe that we can apply similar proofs to different knot invariants. In particular, it is known that the colored Jones polynomial can be obtained from taking a sum of Jones polynomials on different cablings of a knot diagram [27]. In this sense, every colored Jones polynomial corresponds to a finite number of link diagrams in which, possibly using the same construction in Theorem 7 and 8, may give similar computational complexity results.

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