# Computing link polynomials using graphs

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## Introduction

Throughout history, knots have been designated many meanings and uses. In the 1800s, knots were believed to have deep connections to the physical world, which initiated the study of knots for scientific purposes. With the rise of topology, mathematicians began investigating knots which led to many powerful techniques in pure mathematics as well as applications in applied fields such as biology and quantum physics. Here, we investigate a fundamental question of knot theory and present new expansions of previous results to better understand knots and their properties.

## Motivation

We say that a *knot* is a smooth embedding $S^1 \to S^3$. Similarly, a *link* is a disjoint collection of smooth embeddings $S^1 \to S^3$, which may be entangled.

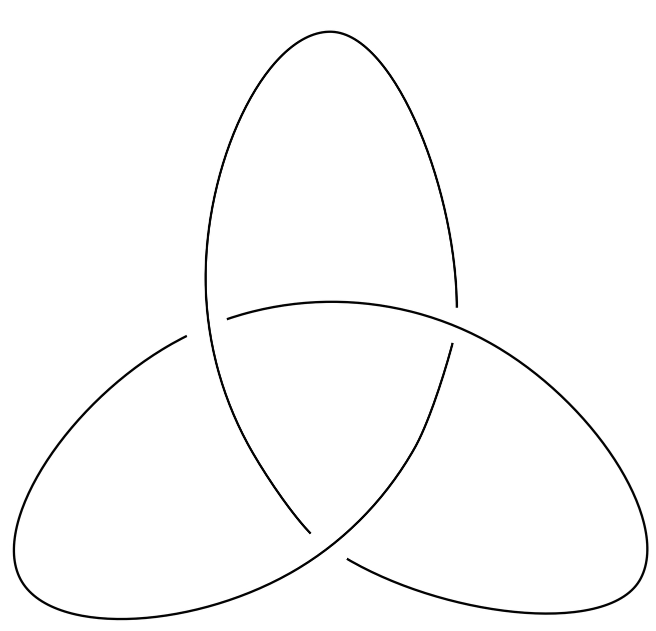

Figure 1: the trefoil knot

We say that links are equivalent if we can continuously deform one link into the other, without ripping or gluing. In general, finding such a deformation is challenging. Further, proving that such a deformation doesn't exist is even harder. To help attack this problem, mathematicians developed *link invariants*. Links that are different with respect to some invariant are not equivalent; however, links with the same invariant are not necessarily equivalent. In this sense, link invariants give us a coarse measure of equivalence. Computing link invariants in general can be very complicated, and it is of great interest to find easier ways to calculate them.

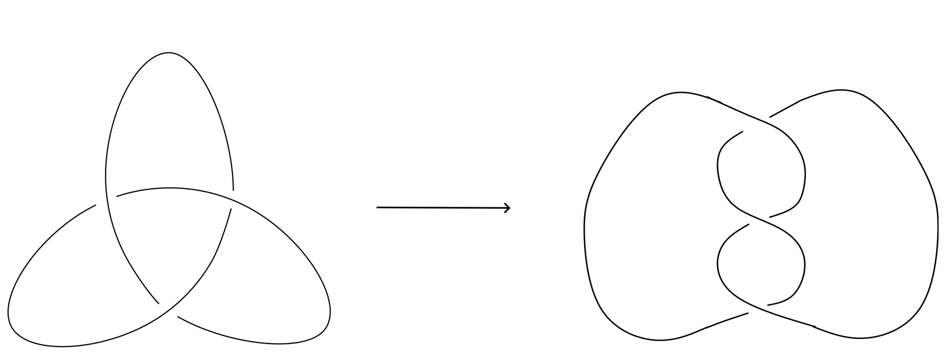

Figure 2: The trefoil is $\sigma_1^3 \in B_2$

## Background

One type of link invariant is called a *link polynomial*, where links are assigned some polynomial. A natural approach to computing these link polynomials is through the use of graph theory. A *graph* is a set of vertices which are connected by edges. A graph is *weighted* if its edges are assigned a value. If we color edges of a graph such that every vertex is covered exactly once, then we have a *perfect matching*.

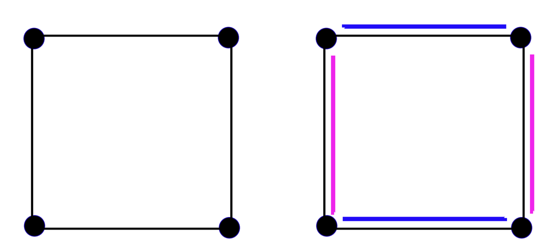

Figure 3: A graph G and its 2 perfect matchings

A *spanning tree* is the smallest subgraph such that every vertex of a graph is still covered by an edge.

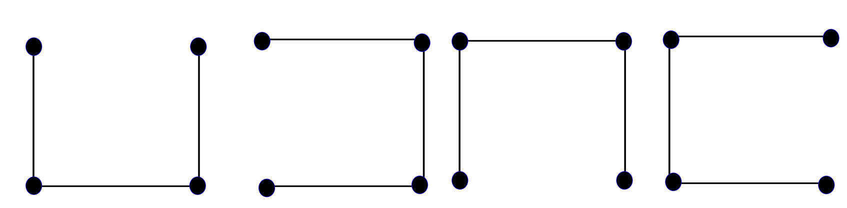

Figure 4: all spanning trees of G

Every link has a corresponding graph called a Tait graph. By assigning letters as weights to a link's Tait graph, we can compute its link polynomials. This method, which was first used by Moshe Cohen, is computationally faster than the standard way to compute these invariants. We have applied this method to a different class of links coming from closed braids.

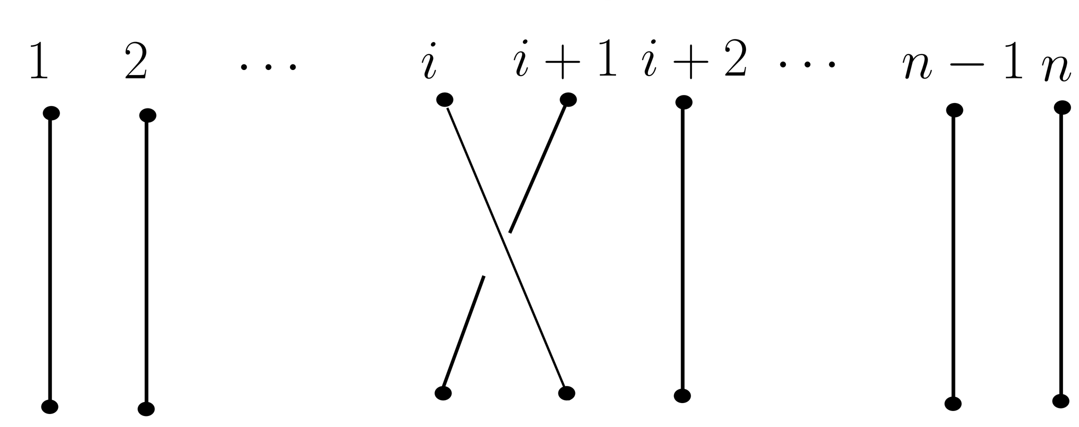


Figure 5: $\sigma_i \in B_n$

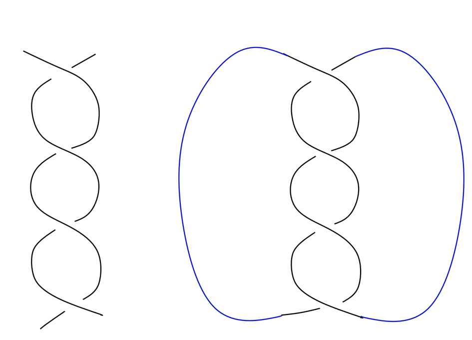

Figure 6: $\sigma_1^4 \in B_2$ and its closure

## References

[1] Morwen Thistlethwaite.
A spanning tree expansion of the jones polynomial.
*Topology, Vol 26*, 1986.

[2] Moshe Cohen.
*Dimer Models for Knot Polynomials.*
PhD thesis, Louisiana State University, 2010.

## Methods

Computing the determinant of a matrix is computationally fast. This means that it takes relatively few resources for a computer to run an algorithm to calculate the determinant. It is a theorem of Thistlethwaite that the Jones polynomial can be computed by looking at the spanning trees and weights of a link's Tait graph. In general, calculating these spanning trees is computationally demanding. So, we wish to turn this problem of finding spanning trees into taking a determinant.

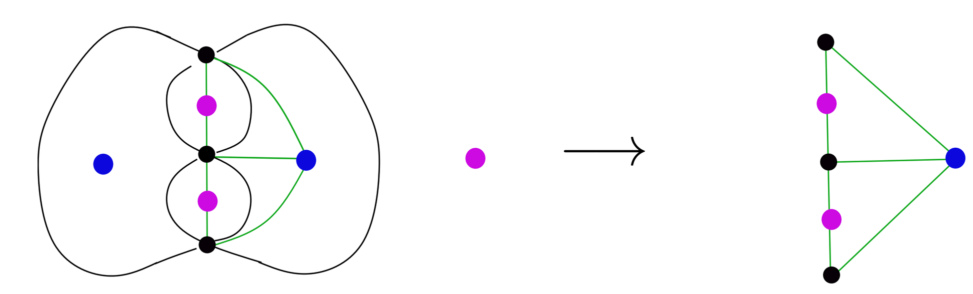

Figure 7: Balanced overlaid Tait graph

By working with the balanced overlaid Tait graph, we find that there are as many perfect matchings as there are spanning trees of the Tait graph. By showing that the weights coming from spanning trees correspond weights from perfect matchings, we can reduce the computation of a link polynomial to a determinant. In particular, we want to show:

$$\sum_{P} \prod_{e \in P} \mu_P(e) = \sum_{T} \prod_{e \in G} \mu_T(e)$$

If this equation above holds for some link $L$ and its diagram, we say that $L$ admits a dimer model.

## Theorem

Let $L$ be a link arising from the closure of an $n$-braid in the form of $\sigma_1^{m_1}\sigma_2^{m_2}\dots\sigma_{n-1}^{m_{n-1}}$ for $|m_i| > 0$. Then $L$ admits a dimer model.

**Corollary** Let $L$ be a $(p, 2)$ torus knot. Then $L$ admits a dimer model.

**Conjecture** Every $(p, q)$ torus knot admits a dimer model.

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