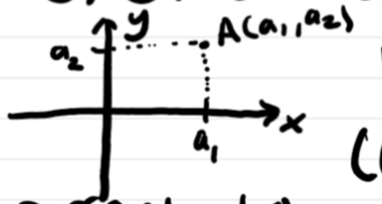
Math 6A Melody melodymolander@math.ucsb.edu

OH: Mondays 4:30-5:30 PDT

Math Lab: Fridays 12-2 PDT

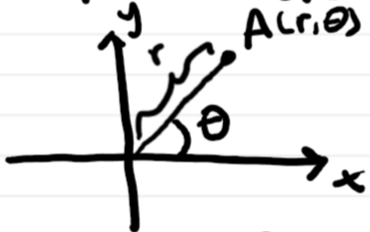
General Math Lab Hours: M-F 9-10, 11-4, 6-8 (PDT)

Vectors xy-plane, Cartesian coordinate system, $\mathbb{R}^2$ ← 2-D



Rectangular Coordinates ← (Cartesian Coordinates)

a_1 & a_2 represent distance from x & y axis



Polar Coordinates ←

r - represents distance from origin

θ - represents angle from the positive x-axis.

To convert from Polar Coordinates to Rectangular: $(r,\theta) \to (x,y)$

$x = r\cos(\theta)$ $y = r\sin(\theta)$ ←

To convert from Rectangular Coordinates to Polar: $(x,y) \to (r,\theta)$

$r = \sqrt{x^2+y^2}$ $\tan\theta = \frac{y}{x}$, $0 \leq \theta < 2\pi$

$$\theta = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } x>0,\ y\geq 0 \\ \arctan\left(\frac{y}{x}\right)+\pi & \text{if } x<0 \\ \arctan\left(\frac{y}{x}\right)+2\pi & \text{if } x>0 \ \&\ y<0 \end{cases}$$

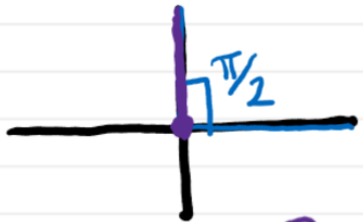
We can generalize Rectangular coordinates to $\mathbb{R}^n$ $(x_1, \dots, x_n)$.

Distance between $a = (a_1, \dots, a_n)$ & $b = (b_1, \dots, b_n)$

$$d(a,b) = \sqrt{(a_1 - b_1)^2 + \dots + (a_n - b_n)^2}$$

Ex: Find the Cartesian Coordinates whose polar coordinates are (i) $(0, \pi/2)$ (ii) $(2, 3\pi/4)$

(i) $(0, \frac{\pi}{2})$ with $r = 0$, $\theta = \frac{\pi}{2}$



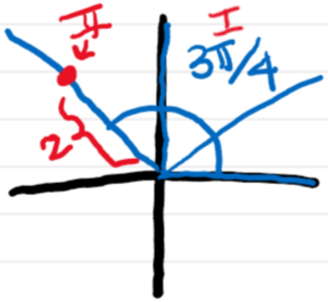
$x = r\cos(\theta)$ $\quad$ $y = r\sin(\theta)$

$x = 0 \cdot \cos(\frac{\pi}{2})$ $\quad$ $y = 0 \cdot \sin(\frac{\pi}{2})$

$x = 0$ $\quad$ $y = 0$

$\boxed{(0,0)}$

(ii) $(2, \frac{3\pi}{4})$ with $r = 2$, $\theta = \frac{3\pi}{4}$



$x = r\cos\theta$ $\quad$ $y = r\sin\theta$

$x = 2\cos(\frac{3\pi}{4})$ $\quad$ $y = 2\sin(\frac{3\pi}{4})$

$x = 2\left(-\frac{\sqrt{2}}{2}\right)$ $\quad$ $y = 2\left(\frac{\sqrt{2}}{2}\right)$

$x = -\sqrt{2}$ $\quad$ $y = \sqrt{2}$

$\boxed{(-\sqrt{2}, \sqrt{2})}$

An <u>n-dimensional vector</u> is an n-tuple $\vec{v}=(v_1,\dots,v_n)$ of real numbers. We can visualize a 2-D vector as a directed line segment joining the origin to (v_1,v_2)

We can generalize this idea in $\mathbb{R}^n$.

The <u>length</u> of vector $\vec{v}=(v_1,\cdots,v_n)$

$$\|\vec{v}\|=\sqrt{v_1^2+\cdots+v_n^2}$$

We can add vectors $\vec{v}=(v_1,\dots,v_n)$ & $\vec{w}=(w_1,\dots,w_n)$

$$\vec{v}+\vec{w}=(v_1+w_1,\cdots,v_n+w_n)$$

We can multiply vectors by a scalar (a number)

$$\alpha\vec{v}=(\alpha v_1,\cdots,\alpha v_n)$$

<u>Ex</u> Find the length of $\vec{v}=(0,2,-1)$

$$\|\vec{v}\|=\sqrt{0^2+2^2+(-1)^2}=\boxed{\sqrt{5}}$$

if $\alpha>0$ then $\alpha\vec{v}$ & $\vec{v}$ have the same direction

A vector whose length is 1 is a unit vector
If $\vec{v}$ is a nonzero vector then $\frac{\vec{v}}{\|\vec{v}\|}$ is the unit vector in the direction of $\vec{v}$. Constructing $\frac{\vec{v}}{\|\vec{v}\|}$ is called normalizing $\vec{v}$

$\vec{v} = (v_1, v_2, v_3)$ can also be written as

$$\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$$

where $\hat{\imath} = (1,0,0)$ $\hat{\jmath} = (0,1,0)$ & $\hat{k} = (0,0,1)$

Ex Let $\vec{a} = 2\hat{\imath} - \hat{\jmath} + 1\hat{k}$ & $\vec{b} = 1\hat{k} - 3\hat{\imath}$. (i) Normalize $\vec{a}$ (ii) Compute $\vec{a} - 2\vec{b}$ (iii) Find the unit vector in the direction of $\vec{a} - 2\vec{b}$

(i) $\frac{\vec{a}}{\|\vec{a}\|}$ $\quad \|\vec{a}\| = \sqrt{2^2 + (-1)^2 + (1)^2} = \sqrt{6}$

$$\frac{\vec{a}}{\|\vec{a}\|} = \boxed{\frac{2}{\sqrt{6}}\hat{\imath} - \frac{1}{\sqrt{6}}\hat{\jmath} + \frac{1}{\sqrt{6}}\hat{k}} \quad \boxed{\left(\frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)}$$

(ii) $\vec{a} - 2\vec{b} = (2 - 2\cdot(-3))\hat{\imath} + (-1 - 2\cdot 0)\hat{\jmath} + (1 - 2\cdot 1)\hat{k}$

$$= \boxed{8\hat{\imath} - \hat{\jmath} - \hat{k}}$$

(iii) $\frac{\vec{a} - 2\vec{b}}{\|\vec{a} - 2\vec{b}\|}$ $\quad \|\vec{a} - 2\vec{b}\| = \sqrt{8^2 + (-1)^2 + (-1)^2} = \sqrt{66}$

$$\frac{\vec{a} - 2\vec{b}}{\|\vec{a} - 2\vec{b}\|} = \boxed{\frac{8}{\sqrt{66}}\hat{\imath} - \frac{1}{\sqrt{66}}\hat{\jmath} - \frac{1}{\sqrt{66}}\hat{k}}$$

Ex Find two vectors $\vec{v}$ & $\vec{w}$ in $\mathbb{R}^2$ such that

$$\|\overrightarrow{v-w}\| = \|\vec{v}\| - \|\vec{w}\|$$

$$\vec{v} = (v_1, v_2) \qquad \vec{w} = (w_1, w_2) \qquad \overrightarrow{v-w} = (v_1-w_1, v_2-w_2)$$

$$\|\overrightarrow{v-w}\| = \sqrt{(v_1-w_1)^2 + (v_2-w_2)^2} \quad \|\vec{v}\| = \sqrt{v_1^2+v_2^2} \quad \|\vec{w}\| = \sqrt{w_1^2+w_2^2}$$

$$\left(\sqrt{(v_1-w_1)^2+(v_2-w_2)^2}\right)^2 = \left(\sqrt{v_1^2+v_2^2} - \sqrt{w_1^2+w_2^2}\right)^2 = \left(\sqrt{v_1^2+v_2^2} - \sqrt{w_1^2+w_2^2}\right)\left(\sqrt{v_1^2+v_2^2} - \sqrt{w_1^2+w_2^2}\right)$$

$$(v_1-w_1)^2 + (v_2-w_2)^2 = v_1^2+v_2^2 - 2\sqrt{v_1^2+v_2^2}\sqrt{w_1^2+w_2^2} + w_1^2+w_2^2$$

$$\cancel{v_1^2} - 2v_1w_1 + \cancel{w_1^2} + \cancel{v_2^2} - 2v_2w_2 + \cancel{w_2^2} = \cancel{v_1^2} + \cancel{v_2^2} - 2\sqrt{(v_1^2+v_2^2)(w_1^2+w_2^2)} + \cancel{w_1^2} + \cancel{w_2^2}$$

$$\cancel{-2}v_1w_1 \cancel{-2}v_2w_2 = \cancel{-2}\sqrt{(v_1^2+v_2^2)(w_1^2+w_2^2)}$$

$$(v_1w_1 + v_2w_2)^2 = \left(\sqrt{(v_1^2+v_2^2)(w_1^2+w_2^2)}\right)^2$$

$$v_1^2w_1^2 + 2v_1w_1v_2w_2 + v_2^2w_2^2 = (v_1^2+v_2^2)(w_1^2+w_2^2)$$

$$\cancel{v_1^2w_1^2} + 2v_1w_1v_2w_2 + \cancel{v_2^2w_2^2} = \cancel{v_1^2w_1^2} + v_1^2w_2^2 + v_2^2w_1^2 + \cancel{v_2^2w_2^2}$$

$$2v_1w_1v_2w_2 = v_1^2w_2^2 + v_2^2w_1^2$$

$$0 = v_1^2w_2^2 - 2v_1w_1v_2w_2 + v_2^2w_1^2$$

$$0 = (v_1w_2 - v_2w_1)^2$$

$$\boxed{v_1w_2 = v_2w_1}$$

$$\vec{v} = (v_1, v_2) \qquad \vec{w} = (w_1, w_2)$$

$$\|\overrightarrow{v-w}\| = \|\vec{v}\| - \|\vec{w}\|$$

e.g. $\vec{v} = (4, 8) \quad \vec{w} = (2, 4)$

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