Math Lab: M-F 12-5, 6-8 PDT

My hours: F 12-2

Notes & Video

web.math.ucsb.edu/~melodymolander/math6a_spring2020.html

Recall from Calc I $f:\mathbb{R}\to\mathbb{R}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$f:\mathbb{R}^{\overset{\text{input}}{m}} \to \mathbb{R}^{\overset{\text{output}}{n}}$

$f(\underset{\text{real}}{x_1},\dots,\underset{\text{real}}{x_m}) = (\underset{\text{real}}{y_1},\dots,\underset{\text{real}}{y_n})$

We now can take partial derivatives

$f:\mathbb{R}^2\to\mathbb{R}$

$$f_x = \frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

Partial derivatives work the same way except you claim everything except x is a constant (for f_x)

Ex Compute $\frac{\partial f}{\partial y}(2,0) = f_y(2,0)$ & $\frac{\partial f}{\partial x} = f_x$ for

$$f(x,y) = (x-3)^2 e^y$$

$$\frac{\partial f}{\partial y} = (x-3)^2 e^y$$

$\frac{d}{dy} Ce^y = Ce^y$

$$\frac{\partial f}{\partial y} = (x-3)^2 e^y \qquad \frac{d}{dy} Ce^y = Ce^y$$

$$\frac{\partial f}{\partial y}(\underset{x}{2},\underset{y}{0}) = (2-3)^2 e^0 = (-1)^2 \cdot 1 = 1$$

$$\frac{\partial f}{\partial x} = 2(x-3)^1 \cdot 1 \cdot e^y \qquad (x-3)^2 \cdot e^c \sim (x-3)^2 \cdot C$$

$$= \boxed{2(x-3)e^y}$$

Let $f: \mathbb{R}^m \to \mathbb{R}^1 = \mathbb{R}$.

$f(x_1, \dots, x_m)$ gives a single real value

Say $\vec{x} = (x_1, \dots, x_m)$ is the input.

Then

$$Df(\vec{x}) = \nabla f(\vec{x}) = [f_{x_1}(\vec{x}), \dots, f_{x_m}(\vec{x})] \leftarrow \text{vector}$$

this is called the gradient of f at $\vec{x}$

Let $f: \mathbb{R}^m \to \mathbb{R}^n$

$$f(x_1, \dots, x_m) = (f_1(x_1, \dots, x_m), \dots, f_n(x_1, \dots, x_m))$$

e.g. $f: \mathbb{R}^2 \to \mathbb{R}^3$

$$f(x,y) = (xy, x-y, x^2)$$

$$f_1(x,y) = x_2 \quad f_2(x,y) = x-y \quad f_3(x,y) = x^2$$

then

$$Df(\vec{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_m} \end{bmatrix} \leftarrow \nabla f \text{ if } f: \mathbb{R}^m \to \mathbb{R}$$

So when $n=1$ $f: \mathbb{R}^m \to \mathbb{R}$ $\nabla f = Df$

Why is the gradient important?

$\nabla f(\vec{x})$ gives the direction of the largest rate of increase in f at $\vec{x}$

Ex Let $f(x,y,z) = x^2+y^2+z^2$. In what direction does f increase most rapidly at the point

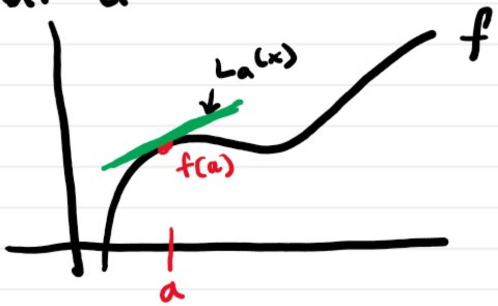
$(1,2,-1)$?

$$\nabla f(1,2,-1) = [f_x(1,2,-1), f_y(1,2,-1), f_z(1,2,-1)]$$

$f_x = 2x$ $\quad f_x(1,2,-1) = 2$
$f_y = 2y$ $\quad f_y(1,2,-1) = 4$
$f_z = 2z$ $\quad f_z(1,2,-1) = -2$

$$\nabla f(1,2,-1) = \boxed{[2,4,-2]}$$

Recall from Calc I linear approximation of $f: \mathbb{R} \to \mathbb{R}$ at a



Near a, $L_a(x)$ is really close to f

$$L_a(x) = f(a) + f'(a)(x-a)$$

We can generalize this for multivariable functions.

$$L_{\vec{a}}(\vec{x}) = f(\vec{a}) + \underline{Df(\vec{a})}(\vec{x}-\vec{a})$$

Ex Find $L_{(1,1)}(\vec{x})$ when $f(x,y) = \underline{1-x^2-2}y^2$

$f: \mathbb{R}^2 \to \mathbb{R}^1 = \mathbb{R}$

$Df = \nabla f$

$$L_{(1,1)}(\vec{x}) = \underline{f(1,1)} + \underline{\nabla f(1,1)}\,(\underline{\vec{x}-(1,1)})$$

- $f(1,1) = 1-1^2-2\cdot 1^2 = -2$
- $\nabla f(1,1) = [f_x(1,1), f_y(1,1)]$

$f_x = -2x \qquad f_x(1,1) = -2$

$f_y = -4y \qquad f_y(1,1) = -4$

$\nabla f(1,1) = [-2,-4]$

- $(\vec{x}-(1,1)) = (x,y)-(1,1) = (x-1, y-1) = \begin{bmatrix} x-1 \\ y-1 \end{bmatrix}$

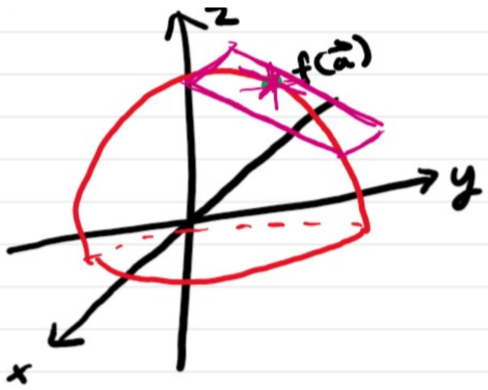
$$L_{(1,1)}(\vec{x}) = -2 + [-2,-4]\begin{bmatrix} x-1 \\ y-1 \end{bmatrix}$$

$$= -2 + -2(x-1) - 4(y-1)$$

$$= -2 - 2x + 2 - 4y + 4$$

$$\boxed{L_{(1,1)}(x) = 4 - 2x - 4y}$$

A linear approximation represents the equation of a plane in $\mathbb{R}^3$. So the equation of the tangent plane is the same as the linear approximation.



<u>Ex</u> Compute the equation of the plane tangent to $f(x,y) = \arctan(xy)$ at $(1,1)$

$f: \mathbb{R}^2 \to \mathbb{R}$ $\qquad Df = \nabla f$

$$L_{(1,1)}(\vec{x}) = f(1,1) + \nabla f(1,1)(\vec{x} - (1,1))$$

- $f(1,1) = \arctan(1) = \frac{\pi}{4}$
- $\nabla f(1,1) = [f_x(1,1), f_y(1,1)]$

$$f_x = \frac{y}{1+x^2y^2} \qquad f_x(1,1) = \frac{1}{1+1^2\cdot 1^2} = \frac{1}{2}$$

$$f_y = \frac{x}{1+x^2y^2} \qquad f_y(1,1) = \frac{1}{1+1^2\cdot 1^2} = \frac{1}{2}$$

$$\nabla f(1,1) = \left[\tfrac{1}{2}, \tfrac{1}{2}\right]$$

- $\vec{x} - (1,1) = (x,y) - (1,1) = (x-1, y-1) = \begin{bmatrix} x-1 \\ y-1 \end{bmatrix}$

$$L_{(1,1)}(\vec{x}) = \frac{\pi}{4} + \left[\tfrac{1}{2}, \tfrac{1}{2}\right]\begin{bmatrix} x-1 \\ y-1 \end{bmatrix}$$

$$z = \frac{\pi}{4} + \frac{1}{2}(x-1) + \frac{1}{2}(y-1)$$

$$0 = \frac{\pi}{4} + \frac{1}{2}(x-1) + \frac{1}{2}(y-1) - z$$

$$0 = \frac{\pi}{4} + \frac{1}{2}x - \frac{1}{2} + \frac{1}{2}y - \frac{1}{2} - 2$$

$$\left[0 = \left(\frac{\pi}{4} - 1\right) + \frac{1}{2}x + \frac{1}{2}y - 2\right] \times 4$$

$$\boxed{0 = (\pi - 4) + 2x + 2y - 4z}$$

Ex Let $f(x,y) = (xy,\ x-y,\ x^2)$. Find $L_{(1,2)}(\vec{x})$

$f: \mathbb{R}^2 \to \mathbb{R}^3$

$$L_{(1,2)}(\vec{x}) = f(1,2) + Df(1,2)(\vec{x} - (1,2))$$

$$f(1,2) = (2, -1, 1)$$

$Df(1,2)$

$$f(x,y) = (xy,\ x-y,\ x^2)$$

$$f_1(x,y) = xy \quad f_2(x,y) = x-y \quad f_3(x,y) = x^2$$

$$Df(1,2) = \begin{bmatrix} \partial f_1/\partial x\,(1,2) & \partial f_1/\partial y\,(1,2) \\ \partial f_2/\partial x\,(1,2) & \partial f_2/\partial y\,(1,2) \\ \partial f_3/\partial x\,(1,2) & \partial f_3/\partial y\,(1,2) \end{bmatrix}$$

$$\frac{\partial f_1}{\partial x} = y \quad \frac{\partial f_1}{\partial x}(1,2) = 2 \qquad \frac{\partial f_1}{\partial y} = x \quad \frac{\partial f_1}{\partial y}(1,2) = 1$$

$$\frac{\partial f_2}{\partial x} = 1 \qquad \frac{\partial f_2}{\partial y} = -1 \qquad \frac{\partial f_3}{\partial x} = 2x \quad \frac{\partial f_3}{\partial x}(1,2) = 2$$

$$\frac{\partial f_3}{\partial y} = 0$$

$$Df(1,2) = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 2 & 0 \end{bmatrix}$$

$$(\vec{x} - (1,2)) = (x,y) - (1,2) = (x-1,\ y-2) = \begin{bmatrix} x-1 \\ y-2 \end{bmatrix}$$

$$L_{(1,2)}(\vec{x}) = [2,-1,1] + \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x-1 \\ y-2 \end{bmatrix}$$

$$= [2,-1,1] + \begin{bmatrix} 2(x-1)+(y-2) \\ x-1-(y-2) \\ 2(x-1) \end{bmatrix}$$

$$\longleftrightarrow$$

$$= [2,-1,1] + [2(x-1)+(y-2),\ x-1-(y-2),\ 2(x-1)]$$

$$= [2+2(x-1)+(y-2),\ -1+x-1-(y-2),\ 1+2(x-1)]$$

$$= \boxed{[2x+y-2,\ x-y-4,\ 2x]}$$