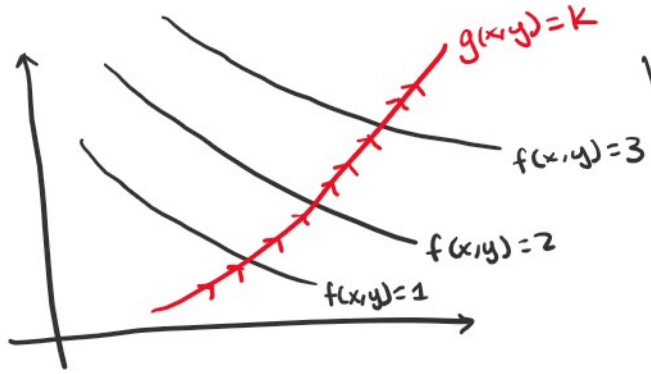
Week 5 7PM

Tuesday, April 28, 2020 6:59 PM

* Midterm: • May 5^{th} during lecture time
  • Submit exam through Gradescope
  • Before exam, join the course on Gradescope!!
  • Announcements PDF on GauchoSpace

We want to find extreme values of a differentiable function f subject to certain conditions, say $g(x,y)=K$



While walking along $g(x,y)=k$ we want to find max & min of f.

LaGrange Multipliers

Let $f:\mathbb{R}^2\to\mathbb{R}^1$, $g:\mathbb{R}^2\to\mathbb{R}^1$ be C^1 functions.
If $f(x,y)$ has a local max or local min subject to constraint $g(x,y)=K$ at $\vec{a}=(x_0,y_0)$ & if $\nabla g(\vec{a})\neq 0$ then $\nabla f(\vec{a})=\lambda\nabla g(\vec{a})$ for some real number λ.

Ex Find the maximum of $V(r,h)=\pi r^2 h$ subject to $2\pi r^2+2\pi r h=S$ where $\boxed{r,h>0}$ $(S>0)$

Let $g(r,h)=2\pi r^2+2\pi r h$

Step 1: $\boxed{\text{Find } \nabla V \text{ \& } \nabla g \text{ \& make sure } \nabla g\neq 0}$ ✓

$\nabla V=(2\pi r h,\ \pi r^2)$ $\quad\nabla g=(4\pi r+2\pi h,\ 2\pi r)$

$\boxed{r>0}$
$2\pi r\neq 0$ $\Big\}\Rightarrow \nabla g\neq 0$

$\nabla V = \langle 2\pi \ldots \rangle$

$r>0$
$2\pi r \neq 0$
$2\pi r = 0$
$r = 0/2\pi = 0$ $\Rightarrow \nabla g \neq 0$

Step 2: $\boxed{\text{Use } \nabla V = \lambda \nabla g \text{ \& solve for } \lambda}$ ✓

$$\nabla V = \lambda \nabla g$$

$$(2\pi rh, \pi r^2) = \lambda(4\pi r + 2\pi h, 2\pi r)$$

$$2\pi rh = \lambda(4\pi r + 2\pi h) \quad \& \quad \pi r^2 = \lambda 2\pi r$$

$$\lambda = \boxed{\frac{2\pi rh}{4\pi r + 2\pi h}} \quad \& \quad \lambda = \frac{\cancel{\pi} r^{\cancel{2}}}{2\cancel{\pi}\cancel{r}} = \boxed{\frac{r}{2}}$$

Step 3: $\boxed{\text{Set every 2 equations equal \& solve for a variable}}$ ✓

$$\frac{2\pi rh}{4\pi r + 2\pi h} = \frac{r}{2}$$

$$2(2\pi rh) = r(4\pi r + 2\pi h)$$

$$4\pi rh = 4\pi r^2 + 2\pi rh$$

$$4\pi rh - 2\pi rh = 4\pi r^2$$

$$\frac{2\pi rh}{2\pi r} = \frac{{}^2\cancel{4}\cancel{\pi} r^{\cancel{2}}}{\cancel{2}\cancel{\pi}\cancel{r}}$$

$$\boxed{h = 2r} \leftarrow$$

Step 4: $\boxed{\text{Take everything from Step 3 \& plug into constraint \& solve for the variable}}$ ✓

$$2\pi r^2 + 2\pi rh = S$$

$$2\pi r^2 + 2\pi r(2r) = S$$

$$2\pi r^2 + 4\pi r^2 = S$$

$$6\pi r^2 = S$$

$$r^2 = \frac{S}{6\pi}$$

$$r^2 = \frac{S}{6\pi}$$

$$\boxed{r = \pm\sqrt{\frac{S}{6\pi}}} \leftarrow$$

Step 5: Solve for all other variables using equations from Step 3 ✓

$$h = 2r$$

$$\boxed{h = \pm 2\sqrt{\frac{S}{6\pi}}}$$

Step 6: Determine the maximum

or can happen on the boundary

at $\left(\sqrt{\frac{S}{6\pi}}, 2\sqrt{\frac{S}{6\pi}}\right)$ or $\cancel{\left(-\sqrt{\frac{S}{6\pi}}, -2\sqrt{\frac{S}{6\pi}}\right)}$

$r, h > 0$

Boundary: $2\pi r^2 + 2\pi r h = S \leftarrow$ hyperbola No bounds

$$V(r,h) = \pi r^2 h$$

$$V\left(\sqrt{\frac{S}{6\pi}}, 2\sqrt{\frac{S}{6\pi}}\right) = \pi\left(\sqrt{\frac{S}{6\pi}}\right)^2 \cdot 2\sqrt{\frac{S}{6\pi}} = \cancel{\pi}\left(\frac{S}{\cancel{6\pi}_{3}}\right)\cancel{2}\sqrt{\frac{S}{6\pi}}$$

$$= \boxed{\frac{S}{3}\sqrt{\frac{S}{6\pi}}}$$

Max happens at $\left(\sqrt{\frac{S}{6\pi}}, 2\sqrt{\frac{S}{6\pi}}\right)$

& Max is $\frac{S}{3}\sqrt{\frac{S}{6\pi}}$ □

Let $\vec{F} = (F_1, F_2, F_3) : \mathbb{R}^3 \to \mathbb{R}^3$ be a differentiable vector field. The divergence of $\vec{F}$ is a function vector field. The divergence of $\vec{F}$ is a

$$\boxed{\operatorname{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}}$$

The curl of $\vec{F}$ is the vector field:

$$\boxed{\operatorname{curl}\vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\imath} - \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}}$$

Ex Let $\vec{F}(x,y,z) = (\underset{F_1}{x^3}, \underset{F_2}{xy}, \underset{F_3}{e^{xyz}})$. Compute $\operatorname{div}\vec{F}$ & $\operatorname{curl}\vec{F}$.

$$\boxed{\operatorname{div}\vec{F} = 3x^2 + x + xye^{xyz}}$$

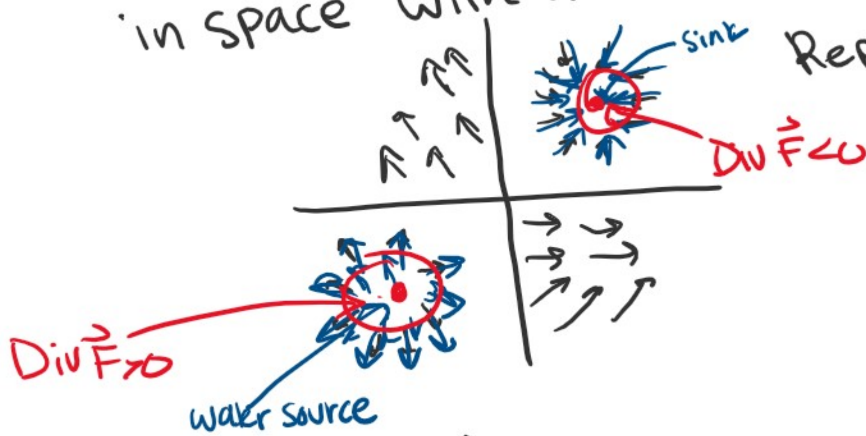
$$\operatorname{curl}\vec{F} = (xze^{xyz} - 0)\hat{\imath} - (0 - yze^{xyz})\hat{\jmath} + (y - 0)\hat{k}$$

$$\boxed{\operatorname{curl}\vec{F} = xze^{xyz}\hat{\imath} + yze^{xyz}\hat{\jmath} + y\hat{k}}$$

What is divergence & curl?

Youtube: 3Blue1Brown & Divergence

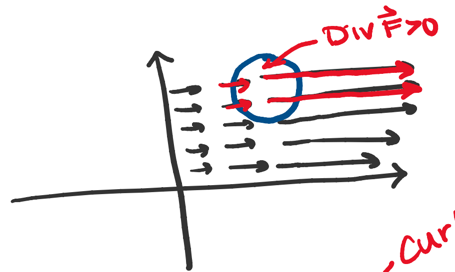
A vector field is associating each point in space with a vector.



Represent:
- magnetic field
- force of gravity
- fluid flow

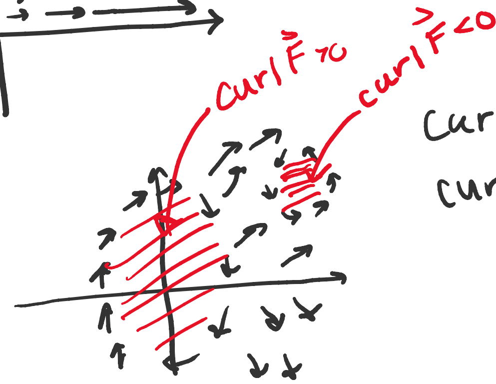
Div $\vec{F}(x,y)$ - measures how much (x,y) "generates" fluid

Div $\vec{F}(x,y)$ - measures how much (x,y) "generates" fluid



curl $\vec{F}(x,y)$

how much fluid rotates around that point?



curl $\vec{F}>0$ clockwise

curl $\vec{F}<0$ counter-clockwise

e.g. curl $\vec{F}(1,1) = 1000$

curl $\vec{F}(-1,1) = 0.1$

curl $\vec{F}(-1,-1) = -1000$

curl $\vec{F}(1,-1) = -0.1$