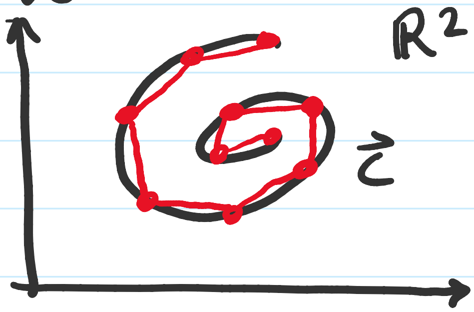
Week 6 7PM

Tuesday, May 5, 2020 6:59 PM

We want to figure out a way to find a length of a curve



Let $\vec{c} : [a,b] \to \mathbb{R}^2$ be a C^1 path. The length $\ell(\vec{c})$ of $\vec{c}$ is given by

$$\ell(\vec{c}) = \int_a^b \|\nabla c\| \, dt$$

__Example__ Compute the length of the path $\vec{c}(t) = (\cos(t), \sin(t))$ $t \in [0, 2\pi]$

$$\ell(\vec{c}) = \int_0^{2\pi} \|\underline{\nabla c}\| \, dt$$

$$\nabla c = (-\sin(t), \cos(t))$$

$$\|\nabla c\| = \sqrt{(-\sin(t))^2 + (\cos(t))^2} = \sqrt{\sin^2(t) + \cos^2(t)} = \sqrt{1} = 1$$

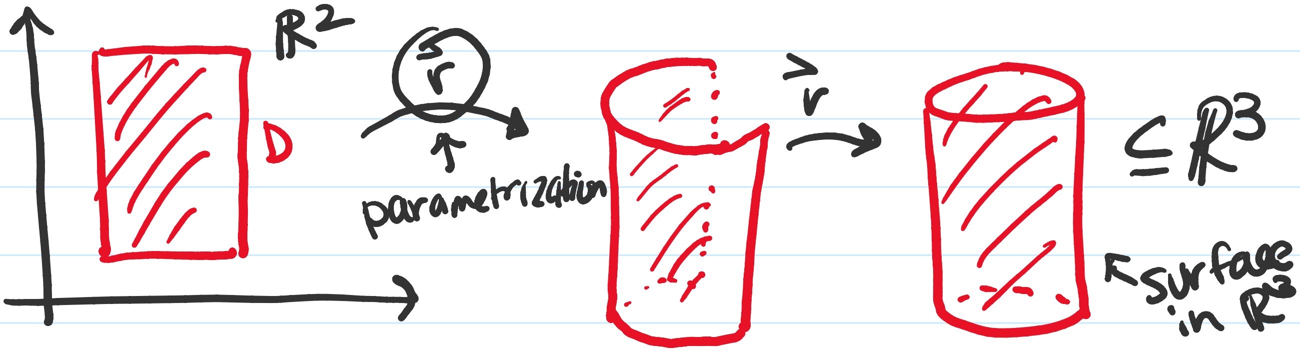
$\sin^2(t) + \cos^2(t) = 1$

$$\ell(\vec{c}) = \int_0^{2\pi} 1 \, dt = t \Big|_0^{2\pi} = 2\pi - 0 = \boxed{2\pi}$$

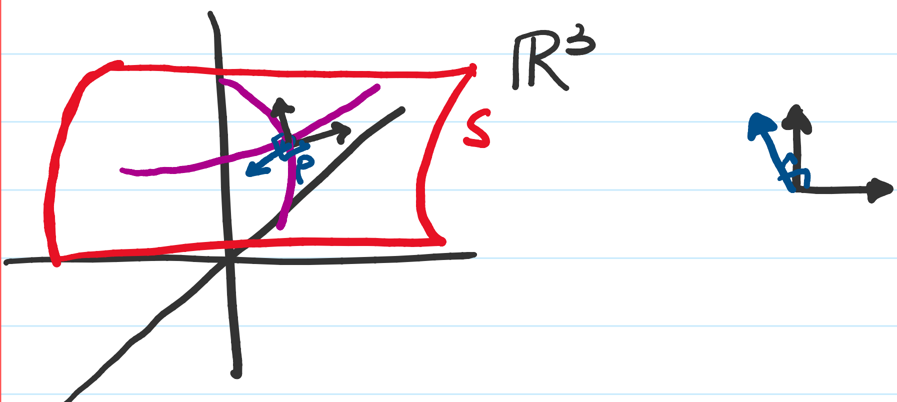
A curve $\vec{c}(t) = (x(t), y(t), z(t))$ $t \in [a,b]$

A curve $\vec{c}(t) = (x(t), y(t), z(t))$ $\quad t \in [a,b]$ is called a <u>parametric representation</u>

A <u>parametrized surface</u> is a map $\vec{r}: D \to \mathbb{R}^3$ where D is a region of $\mathbb{R}^2$ & $\vec{r}$ is one-to-one except possibly on the boundary. The image $S = \vec{r}(D)$ is called a <u>surface</u>



A vector $\vec{v}$ is <u>tangent</u> to the surface S at the point p if $\vec{v}$ is a tangent vector at p for some curve contained in S



If a surface is represented by
$$\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$$
then the tangent vector T_u to a curve C_{v_0} at $r(u_0, v_0)$ is given by

$$\longrightarrow T_u(u_0,v_0) = \left(\frac{\partial x}{\partial u}\bigg|_{(u_0,v_0)}, \frac{\partial y}{\partial u}\bigg|_{(u_0,v_0)}, \frac{\partial z}{\partial u}\bigg|_{(u_0,v_0)} \right)$$

$\rightarrow T_u(u_0,v_0) = \left(\frac{\partial x}{\partial u}\Big|_{(u_0,v_0)}, \frac{\partial y}{\partial u}\Big|_{(u_0,v_0)}, \frac{\partial z}{\partial u}\Big|_{(u_0,v_0)} \right)$

The vector $\vec{N} = T_u \times T_v$ is perpendicular to both T_u & T_v & is called the <u>normal vector</u>.

$$\vec{N} = T_u \times T_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial x/\partial u & \partial y/\partial u & \partial z/\partial u \\ \partial x/\partial v & \partial y/\partial v & \partial z/\partial v \end{vmatrix}$$

<u>Example</u> Consider the cone

$\rightarrow r(u,v) = v\cos(u)\hat{i} + v\sin(u)\hat{j} + v\hat{k}$

$0 \le u \le 2\pi$ & $0 \le v \le 6$. Find T_u, T_v, N & the equation of the tangent plane at (π/π)

$T_u = (\partial x/\partial u, \partial y/\partial u, \partial z/\partial u)$

$\frac{\partial x}{\partial u} = -v\sin(u)$ $\quad \frac{\partial y}{\partial u} = v\cos(u)$ $\quad \frac{\partial z}{\partial u} = 0$

$T_u(\pi,\pi) = (-\pi\sin(\pi), \pi\cos(\pi), 0)$
$= (0, -\pi, 0) \leftarrow$

$T_v = (\partial x/\partial v, \partial y/\partial v, \partial z/\partial v)$

$\frac{\partial x}{\partial v} = \cos(u)$ $\quad \frac{\partial y}{\partial u} = \sin(u)$ $\quad \frac{\partial z}{\partial u} = 1$

$T_v(\pi,\pi) = (\cos(\pi), \sin(\pi), 1) = (-1, 0, 1)$

$\vec{N} = T_u \times T_v$

$\vec{N}(\pi,\pi) = \Big| \ \hat{i} \quad \hat{j} \quad \hat{k} \ \Big|$

$$\vec{N}(\pi,\pi) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -\pi & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

$$= -\pi\hat{i} - 0\hat{j} - \pi\hat{k}$$

$$\boxed{\vec{N}(\pi,\pi) = -\pi\hat{i} - \pi\hat{k}}$$

Equation of a tangent plane that contains $r(u_0,v_0)$ & is Spanned $T_u(u_0,v_0)$ & $T_v(u_0,v_0)$ is:

$$\rightarrow \underline{a}(x-x_0) + \underline{b}(y-y_0) + \underline{c}(z-z_0) = 0 \leftarrow$$

where $N(u_0,v_0) = (a, b, c)$

$r(u_0,v_0) = (x_0, y_0, z_0)$

$N(\pi,\pi) = (-\pi, 0, -\pi)$

$$r(u,v) = v\cos(u)\hat{i} + v\sin(u)\hat{j} + v\hat{k}$$

$$r(\pi,\pi) = \pi\cos(\pi)\hat{i} + \pi\sin(\pi)\hat{j} + \pi\hat{k}$$

$$= -\pi\hat{i} + 0\hat{j} + \pi\hat{k}$$

$$= (-\pi, 0, \pi)$$

$$-\pi(x+\pi) - 0(y-0) - \pi(z-\pi) = 0 \leftarrow$$

$$ax + by + cz = d \leftarrow$$

$$-\pi x \cancel{-\pi^2} - 0 - \pi z \cancel{+\pi^2} = 0$$

$$\boxed{\underbrace{-\pi x - \pi z}_{-\pi} = 0} = \boxed{x + z = 0}$$