Week 8 7PM

Tuesday, May 19, 2020 7:01 PM

Let $\vec{F}$ be a vector field. Then if $\int_C \vec{F} \cdot d\vec{s} = 0$ then $\exists$ $f: \mathbb{R}^n \to \mathbb{R}$ where $\vec{F} = \nabla f$

<u>Ex</u> Find a function f so that $\vec{F} = \nabla f$ & $f((1,0)) = 0$ where

$$\vec{F} = \left(\underset{\uparrow \atop \frac{\partial f}{\partial x}}{\frac{x}{x^2+y^2}},\ \underset{\uparrow \atop \frac{\partial f}{\partial y}}{\frac{y}{x^2+y^2}} + y \right)$$

$$\int \frac{\partial f}{\partial x} dx = \int \frac{x}{x^2+y^2} dx = \boxed{\frac{\ln(x^2+y^2)}{2}} + \boxed{\text{constant term}} + \boxed{g(y)} = f(x,y)$$

$$\int \frac{\partial f}{\partial y} dy = \int \left(\frac{y}{x^2+y^2} + y \right) dy = \boxed{\frac{\ln(x^2+y^2)}{2}} + \boxed{\frac{y^2}{2}} + \boxed{\text{constant term}} + \boxed{h(x)} = f(x,$$

$$f(x,y) = \frac{\ln(x^2+y^2)}{2} + c + \frac{y^2}{2}$$

$f(1,0)=0$

$$f(1,0) = \frac{\ln(1^2+0^2)}{2} + c + \frac{0^2}{2} = \frac{\ln(1)}{2} + c = \boxed{c = 0}$$

$$\boxed{f(x,y) = \frac{\ln(x^2+y^2)}{2} + \frac{y^2}{2}}$$

The <u>Work</u> done by a force $\vec{F}$ upon a particle that moves along the trajectory $\vec{c}(t): [a,b] \to \mathbb{R}^3$ is given by $W = \int_a^b \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\, dt$

<u>Ex</u> Compute the work along the curve

Ex Compute the work along the curve $\vec{c}(t) = (\cos(t), \sin(t))$ $t \in [0, \pi/2]$ for

$$\vec{F} = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} + y \right)$$

$$\vec{F}(\vec{c}(t)) = \left(\frac{\cos(t)}{\cos^2(t)+\sin^2(t)}, \frac{\sin(t)}{\cos^2(t)+\sin^2(t)} + \sin(t) \right)$$

$\cos^2(t)+\sin^2(t) = 1$

$$= (\cos(t), \sin(t) + \sin(t))$$

$$= (\cos(t), 2\sin(t))$$

$$\vec{c}\,'(t) = (-\sin(t), \cos(t))$$

$$W = \int_0^{\pi/2} (\cos(t), 2\sin(t)) \cdot (-\sin(t), \cos(t))\, dt$$

$$= \int_0^{\pi/2} (-\cos(t)\sin(t) + 2\sin(t)\cos(t))\, dt$$

$$= \int_0^{\pi/2} \sin(t)\cos(t)\, dt$$

$u = \sin(t)$
$du = \cos(t)\, dt$

$t = 0 \quad u = \sin(0) = 0$
$t = \pi/2 \quad u = \sin(\pi/2) = 1$

$$= \int_{u=0}^{u=1} u\, du$$

$$= \frac{u^2}{2}\Big|_0^1$$

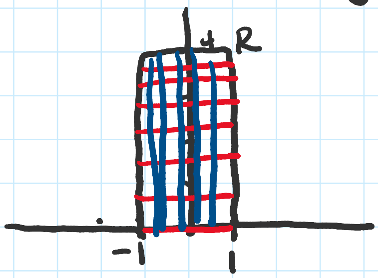
$$= \boxed{1/2}$$

Double Integrals

Approach similarly to partial derivatives
1 variable at a time.

1 variable at a time.

Ex Evaluate $\iint_R 6x^2y\,dA$ where $R=\underbrace{[-1,1]}_{x}\times\underbrace{[0,4]}_{y}$



$-1 \le x \le 1 \qquad 0 \le y \le 4$

$$\int_{x=-1}^{x=1}\left(\int_{y=0}^{y=4} 6x^2y\,dy\right)dx$$

$$= \int_{x=-1}^{x=1} 3x^2y^2 \Big|_{y=0}^{y=4} dx$$

$$= \int_{x=-1}^{x=1} 3x^2\cdot 16\,dx$$

$$= \int_{x=-1}^{x=1} 48x^2\,dx \quad \leftarrow \text{1 integral, 1 variable}$$

$$= 16x^3\Big|_{-1}^{1}$$

$$= 16-(16(-1)^3)$$

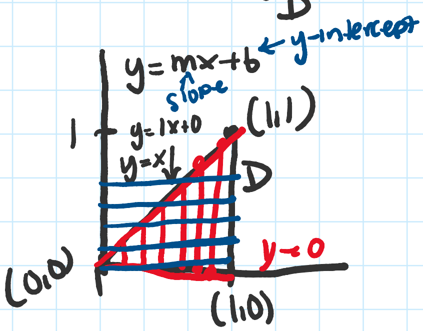
$$= 16+16$$

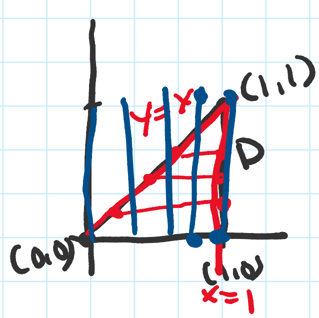
$$= \boxed{32}$$

$16 \times 3 = 48$

Ex Let $T(x,y)=120+30x^2-18y$ on the region D below.

Compute $\iint_D T(x,y)\,dA$





① $a \le x \le b \qquad f(x) \le y \le g(x)$

② $a \le y \le b \qquad f(y) \le x \le f(y)$

② $a \le y \le b \qquad f(y) \le x \le f(y)$

① $0 \le y \le x \qquad 0 \le x \le 1$

② $y \le x \le 1 \qquad 0 \le y \le 1$

① $$\int_{x=0}^{x=1}\left(\int_{y=0}^{y=x} (120 + 30x^2 - 18y)\, dy\right) dx$$

$$= \int_{x=0}^{x=1} 120y + 30x^2y - 9y^2 \Big|_{y=0}^{y=x} dx$$

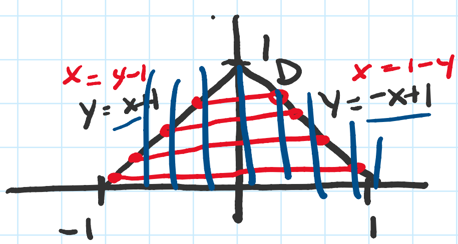
$$= \int_{x=0}^{x=1} 120x + 30x^3 - 9x^2\, dx \quad \leftarrow \text{1 integral \& 1 variable}$$

$$= 60x^2 + \frac{30}{4}x^4 - 3x^3 \Big|_0^1$$

$$= 60 + \frac{30}{4} - 3$$

$$= \boxed{64.5}$$

Ex Evaluate $\iint_D 12 + 2xy^2\, dA$ where D is the region below:



$y-1 \le x \le 1-y \qquad 0 \le y \le 1$

$$\int_{y=0}^{y=1}\left(\int_{x=y-1}^{x=1-y} 12 + 2xy^2\, dx\right) dy$$

$$\int_{y=0} \left(\int_{x=y-1} 12 + 2xy^2 \, dx \right) dy$$

$$= \int_{y=0}^{y=1} 12x + x^2y^2 \Big|_{x=y-1}^{x=1-y} dy$$

$$= \int_{y=0}^{y=1} \left(\left(12(1-y) + (1-y)^2y^2\right) - \left(12(y-1) + (y-1)^2y^2\right) \right) dy$$

$$= \int_{y=0}^{y=1} \left(\left(12 - 12y + (1-2y+y^2)\,y^2\right) - \left(12y - 12 + (y^2-2y+1)\,y^2\right) \right) dy$$

$$= \int_{y=0}^{y=1} \left((12 - 12y + y^2 - 2y^3 + y^4) - (12y - 12 + y^4 - 2y^3 + y^2) \right) dy$$

$$= \int_{y=0}^{y=1} \left(12 - 12y \cancel{+y^2} \cancel{-2y^3} \cancel{+y^4} - 12y + 12 \cancel{-y^4} \cancel{+2y^3} \cancel{-y^2} \right) dy$$

$$= \int_{y=0}^{y=1} (24 - 24y)\, dy \leftarrow$$ 1 integral 1 variable

$$= 24y - 12y^2 \Big|_0^1$$

$$= 24 - 12$$

$$= \boxed{12}$$